What's Fundamental About the Fundamental Theorem?

Bernie Madison, University of Arkansas

Fundamental Theorems

One should take note of theorems that are called fundamental. There are a few such theorems in mathematics including the Fundamental Theorem of Arithmetic, Fundamental Theorem of Algebra, and Fundamental Theorem of Calculus (FTOC).

Do you know the two fundamental theorems other than the FTOC? Can you prove either or both of them and do you know how they are used, i.e. why are they fundamental?

Fundamental Theorem of Arithmetic: Any integer greater than 1 can be expressed as a product of positive primes and this factorization is unique except for the order of the factors.

Fundamental Theorem of Algebra: Every non-constant polynomial with complex coefficients has a complex root.

Our topic here is the FTOC. Why does this theorem rate such a heavy name?

Fundamental Theorem of Calculus (FTOC)

The FTOC is often stated as two theorems as follows, often called first and second or parts 1 and 2. Sometimes one of the two is called a corollary of the other.

FTOC 1. If f is a continuous real-valued function on the interval $[a,b]$ and $a \le c \le b$ then the function defined by $A(x) = \int_c^x f(t)dt$ has a derivative at any x in (a,b) and $A'(x) = f(x)$

FTOC 2. If f is a continuous real-valued function on the interval $[a,b]$, c and d are in the interval $[a,b]$, and A is any antiderivative of f, then $\int_c^d f(x)dx = A(d) - A(c)$.

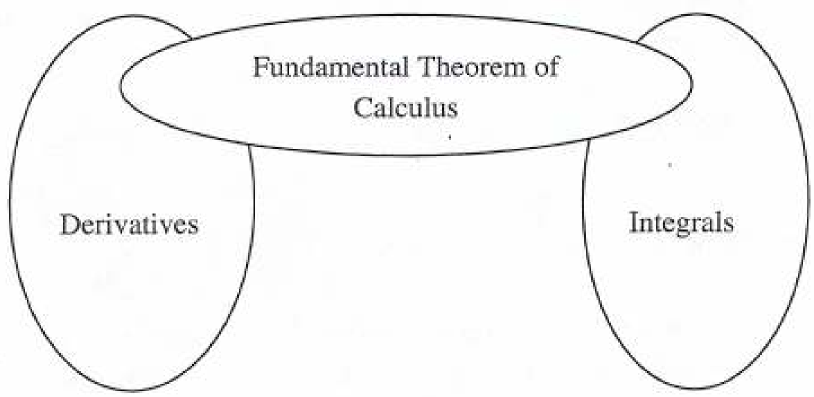
Historical note: The concepts of the derivative and the integral were well known before the time of Newton and Leibniz. However the great English physicist and mathematician, Isaac Newton (1642-1727) and the great German mathematician and philosopher Gottfried Wilhelm Leibniz (1646-1716) were the first to see the i8ntimate relationship between the integral and the derivative stated in the FTOC. Because of this insight, gained independently of each other, they are usually given credit for the discovery of calculus.

Here's another way of expressing the FTOC (from *Calculus with Analytic Geometry* by Johnson and Kiokemeister, 2^{nd} edition, 1960).

FTOC. If f is a function continuous in a closed interval, then:
(1) The function *f* has antiderivatives in this interval.
(2) If *F* is any antiderivative of *f*, and *a* and *b* are numbers in the given interval,

$$\int_a^b F(x)dx = F(b) - F(a).$$

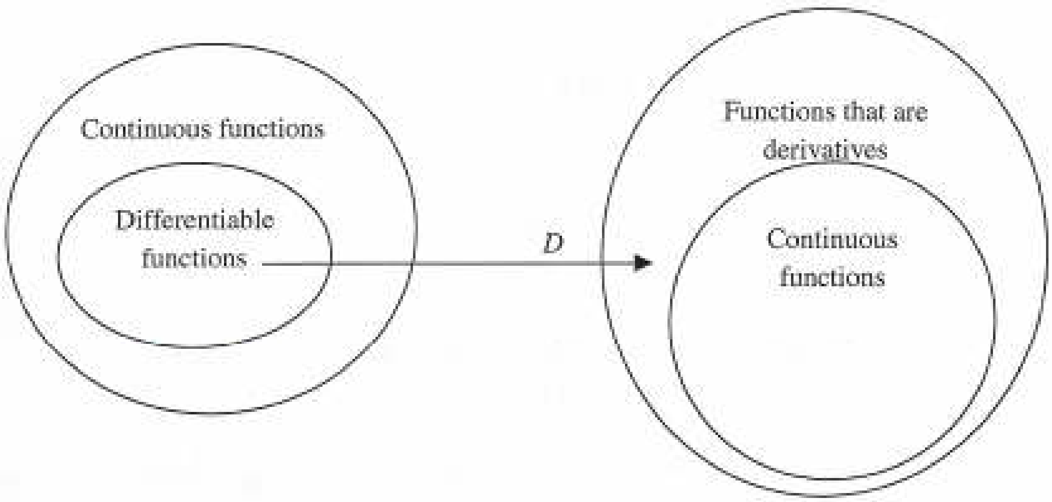
First reason why the FTOC is fundamental: It connects two powerful concepts, both of which have independent value. The connection makes both more powerful. This is illustrated below.



Second reason why the FTOC is fundamental: It is useful in evaluating definite integrals. This is the way it is used most often in calculus courses. Evaluating definite integrals of any function by the definition of the definite integral can be difficult, but using methods of antidifferentiation and FTOC 2 above, the evaluation is simple. We all know how to do this. For example,

$$\int_2^5 x^2 dx = \left[\frac{x^3}{3}\right]_2^5 = \frac{5^3}{3} - \frac{2^3}{3} = \frac{121}{3}.$$

The third reason is based on the following illustration of the operation of differentiation, which I will denote by D.



The above diagram raises some questions about whether the diagram is an accurate illustration of the sets of functions involved.

The FTOC tells us that the set of continuous functions is contained in the set of functions that are derivatives. What about the other containments that are indicated?

(1) Are there continuous functions that are not differentiable?
(2) Are there functions that are not derivatives?
(3) Are there functions that are derivatives that are not continuous?

The first of these is a question in the first course in calculus. There are many examples that can be illustrated early on. For example the absolute value function $f(x)=|x|$ is continuous at any real number but is not differentiable at 0. In fact, there are functions that are continuous at every real number but not differentiable at any real number.

Surely, the answer to the second question is yes, but to look at it and the third question, it helps to learn more about functions that are derivatives.

Intermediate Value Theorem for Derivatives. *Suppose f is differentiable on $[a,b]$, and suppose $f'(a)<c<f'(b)$. Then there is a point $x\in(a,b)$ such that $f'(x)=c$.*
A similar result holds, of course, if $f'(a)>f'(b)$.
Proof. Put

$$g(t) = f(t) - ct \text{ for } a \le t \le b.$$

Then

$$g'(a) = f'(a) - c < 0$$
$$g'(b) = f'(b) - c > 0$$

Let $x \in [a,b]$ be a point at which g attains its minimum. Then show that g does not attain its minimum at either a or b using the fact that $g'(a) < 0$ and $g'(b) > 0$ and these are limits of $\frac{g(t) - g(a)}{t - a}$ and $\frac{g(b) - g(t)}{b - t}$. Hence $a < x < b$. Now using the fact that the derivative of a function where it attains its minimum is 0 we have that $g'(x) = f'(x) - c = 0$ or $f'(x) = c$.

Corollary (Intermediate Value Theorem for Continuous Functions). If f is continuous of $[a,b]$ and $f(a) < c < f(b)$, then there is x, $a < x < b$, so that $f(x) = c$.

Proof. Use FTOC and the IVT for derivatives. IVT for Cont used to prove FTOC

Corollary. Functions that are derivatives cannot have simple discontinuities.

Recall. The function f has a simple discontinuity at a point c provided f is discontinuous at c while both $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist. Examples are removable discontinuities and jump discontinuities.

Hence an example of a function that is not a derivative (question 2) is a simple function such as $f(x) = \begin{cases} -1 \text{ if } x < 0 \\ +1 \text{ if } x \ge 0. \end{cases}$

This result (IVT for derivatives) made answering question 2 easy but made answering question 3 more difficult, but we can do it.

Example 1. $f(x) = \begin{cases} x \sin \frac{1}{x} \text{ if } x \ne 0 \\ 0 \quad \text{ if } x = 0 \end{cases}$

If $x \ne 0$ then $f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$. If $x = 0$ then $f'(0) = \lim_{t \to 0} \frac{f(t) - f(0)}{t - 0} = \lim_{t \to 0} \sin \frac{1}{t}$ which does not exist.

Example 2. $f(x)=\begin{cases} x^2\sin\frac{1}{x} \text{ if } x\neq 0 \\ 0 \quad\quad \text{ if } x=0 \end{cases}$

For $x\neq 0$, $f'(x)=2x\sin\frac{1}{x}-\cos\frac{1}{x}$. For $x=0$, using the definition of the derivative,

$f'(0)=\lim_{t\to 0}\frac{f(t)-f(0)}{t-0}=t\sin\frac{1}{t}$. Since $\left|t\sin\frac{1}{t}\right|\leq|t|$, we have $f'(0)=0$.

Now, f is differentiable at all points x but f' is not continuous at 0 since $\cos\frac{1}{x}$ does not have a limit at 0.

FTOC on the AP Exams

Most occurrences of the FTOC on AP exams in the past (and likely in the future) are one of the following.

1) Evaluation of definite integrals – these are commonplace but sometimes the problem might be stated as follows: Use the FTOC to evaluate this definite integral. That would mean that one should use an antiderivative instead of a numerical process on a calculator.
2) Problems with functions defined by integrals. These are of two basic types:
(a) Given f, define the function g by $g(x)=\int_a^x f(t)dt$. Investigate g.
(b) Given f' and a value $f(x_0)$, find f.

This second type (b) is worth mention because it prompts another statement of the FTOC. We are looking for f, knowing the derivative of f. So we are looking for an antiderivative of f'. Since $\int_{x_0}^x f'(t)dt$ is an antiderivative of f', we know that $\int_{x_0}^x f'(t)dt$ and f' differ by a constant, or $f(x)=\int_{x_0}^x f'(t)dt+C$. Evaluating at x_0 gives $f(x)=\int_{x_0}^x f'(t)dt+f(x_0)$.

This is one way that part of the FTOC is sometimes stated. For example, from *Calculus* (Volume 1) by Apostol (1967), we find the following:

Second Fundamental Theorem of Calculus. *Assume f is continuous on an open interval I, and let P be any primitive (antiderivative) of f on I. Then for each c and each x in I, we have*

$$P(x)=P(c)+\int_c^x f(t)dt$$

Someday, perhaps, more of the above material will be incorporated into the AP exams.