Fooling Newton's Method

a) Find a formula for the Newton sequence, and verify that it converges to a nonzero of f.

$$x_{n+1} = x_n - \frac{1-2x\sin\left(\frac{1}{x}\right)}{\frac{2}{x}\cos\left(\frac{1}{x}\right)-2\sin\left(\frac{1}{x}\right)}$$

$$x_1 = \frac{1}{2\pi}$$

$$x_2 = \frac{1}{2\pi} - \frac{1}{4\pi} = \frac{1}{4\pi}$$

$$x_3 = \frac{1}{4\pi} - \frac{1}{8\pi} = \frac{1}{8\pi}$$

$$x_n = \frac{1}{2^n\pi} \to 0 \text{ } but\ f(0) \neq 0.$$

b) Find a formula for $f'(x_n)$ and determine its behavior as $n \to \infty$.

$$f'(x_n) = \frac{2}{\frac{1}{2^n\pi}}\cos\left(\frac{1}{\frac{1}{2^n\pi}}\right) - 2\sin\left(\frac{1}{\frac{1}{2^n\pi}}\right)$$

$$= 2^{n+1}\pi \to \infty \text{ } as\ n \to \infty$$

A Stirling-like Inequality

Integrate the left and right sides, exponentiate, and complete the inequality:

$$e\cdot\left(\frac{n}{e}\right)^n < n! < e\cdot\left(\frac{n+1}{e}\right)^{n+1}.$$

II. Find the interval of convergence of the power series $\sum_{n=1}^{\infty}\frac{n^n x^n}{n!}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n |x| = e|x|$$

So the radius of convergence is $\frac{1}{e}$.

At $x = \frac{1}{e}$, the series is $\sum_{n=1}^{\infty}\frac{n^n}{e^n n!}$, but $\frac{n^n}{e^n n!} > \frac{n^n}{e^n\left(\frac{n+1}{e}\right)^{n+1} e} = \frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{n+1} > \frac{1}{3}\cdot\frac{1}{n+1}$, by Part I.

So it diverges by comparison.

At $x=-\frac{1}{e}$, we get the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^n n^n}{e^n n!}$. From the real Stirling Inequality, we get that $a_n=\frac{\left(\frac{n}{e}\right)^n}{n!}<\frac{\left(\frac{n}{e}\right)^n}{\left(\frac{n}{e}\right)^n\sqrt{2n\pi}}=\frac{1}{\sqrt{2n\pi}}$, so $a_n=\frac{n^n}{e^n n!}\to 0$. Also

$$a_{n+1}-a_n=\frac{\left(\frac{n+1}{e}\right)^{n+1}}{(n+1)!}-\frac{\left(\frac{n}{e}\right)^n}{n!}=\frac{\left[\left(\frac{n+1}{e}\right)^n-e\left(\frac{n}{e}\right)^n\right]}{en!}=\frac{(n+1)^n-en^n}{e^{n+1}n!}$$

$$=\frac{n^n\left[\left(1+\frac{1}{n}\right)^n-e\right]}{e^{n+1}n!}.$$

But $\left[\left(1+\frac{1}{n}\right)^n-e\right]<0$, so a_n decreases to zero. The Alternating Series Test implies that the series converges at $x=-\frac{1}{e}$. So the interval of convergence is $\left[-\frac{1}{e},\frac{1}{e}\right)$.

III. a) If k is a positive integer, find the radius of convergence of the power series $\sum_{n=0}^{\infty}\frac{(n!)^k}{(kn)!}x^n$.

$\left|\frac{a_{n+1}}{a_n}\right|=\frac{(n+1)^k}{(kn+1)(kn+2)\cdots(kn+k)}|x|\to\frac{|x|}{k^k}$ *as* $n\to\infty$**.** So the radius of convergence is k^k

b) If $k=1$ check the endpoints.

In this case the series is $\sum_{n=0}^{\infty}x^n$, which diverges for $x=\pm 1$.

c) If $k\geq 2$, use the result of I. to check the endpoints.

At $x=k^k$, we get the series $\sum_{n=0}^{\infty}\frac{(n!)^k}{(kn)!}k^{nk}$, but

$$\frac{(n!)^k}{(kn)!}k^{nk} > \frac{\left[e\left(\frac{n}{e}\right)^n\right]^k k^{nk}}{e\left(\frac{kn+1}{e}\right)^{kn+1}} = \frac{en^{nk}}{(kn+1)^{nk}(kn+1)} = e\left(k+\frac{1}{n}\right)^{-nk}\cdot\frac{1}{kn+1}$$

$$= \frac{e}{\left(k^k\right)^n\left[\left(1+\frac{\frac{1}{k}}{n}\right)^n\right]^k}\cdot\frac{1}{kn+1} > \frac{1}{k^k}\cdot\frac{1}{kn+1}$$

From Part I., so it diverges by comparison. We can actually use the Ratio test to determine convergence at both endpoints:

At $x=\pm k^k$, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(kn+k)^k}{(kn+1)(kn+2)\cdots(kn+k)} > 1$, which implies that $|a_{n+1}| > |a_n|$, so $\lim_{n\to\infty}|a_n| \neq 0$, and hence $\lim_{n\to\infty} a_n \neq 0$ and we get divergence at both endpoints.

Evaluating Proper/Improper Integrals with little or no Integration.

I. For the improper integral $\int_0^\infty \frac{\ln x}{1+x^2}dx$

Use the substitution $u=\frac{1}{x}$ to find its value.

$\int_0^\infty \frac{\ln x}{1+x^2}dx = \int_\infty^0 \frac{\ln\left(\frac{1}{u}\right)}{1+\frac{1}{u^2}}\cdot\frac{-1}{u^2}du = -\int_0^\infty \frac{\ln u}{1+u^2}du$, so we can conclude that $\int_0^\infty \frac{\ln x}{1+x^2}dx = 0$.

II. Evaluate $\int_0^\infty \frac{\sqrt{x}\ln x}{(x+1)(x^2+x+1)}dx$ using the substitution $u=\frac{1}{x}$. *{Hint:* $\frac{1}{\sqrt{z}} = \frac{\sqrt{z}}{z}$.*}*

$$\int_0^\infty \frac{\sqrt{x}\ln x}{(x+1)(x^2+x+1)}dx = \int_\infty^0 \frac{\frac{1}{\sqrt{u}}\ln\left(\frac{1}{u}\right)}{\left(\frac{1}{u}+1\right)\left(\frac{1}{u^2}+\frac{1}{u}+1\right)}\cdot\frac{-1}{u^2}du = -\int_0^\infty \frac{\sqrt{u}\ln u}{(u+1)(u^2+u+1)}du,$$ so we can conclude

$$\int_0^\infty \frac{\sqrt{x}\ln x}{(x+1)(x^2+x+1)}dx = 0.$$

III. If you use the substitution $u=\frac{1}{x}$ in the integral $\int_0^\infty \frac{x^2-1}{x^2}dx$, you arrive at $\int_0^\infty \frac{x^2-1}{x^2}dx = \int_\infty^0 \frac{\frac{1}{u^2}-1}{\frac{1}{u^2}}\cdot\frac{-1}{u^2}du = \int_0^\infty \left(\frac{1}{u^2}-1\right)du = -\int_0^\infty \frac{u^2-1}{u^2}du$. Is it okay to conclude that $\int_0^\infty \frac{x^2-1}{x^2}dx = 0$? Explain.

Since the improper integral is divergent, we can't conclude that its value is zero.

IV. a) Use the substitution $u=\frac{\pi}{2}-x$ along with the identities $\sin\left(\frac{\pi}{2}-x\right)=\cos x$ and $\cos\left(\frac{\pi}{2}-x\right)=\sin x$ to evaluate the definite integral $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x+\sin x}dx$.

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x+\sin x}dx = -\int_{\frac{\pi}{2}}^0 \frac{\sin\left(\frac{\pi}{2}-u\right)}{\cos\left(\frac{\pi}{2}-u\right)+\sin\left(\frac{\pi}{2}-u\right)}du = \int_0^{\frac{\pi}{2}} \frac{\cos u}{\sin u+\cos u}du$$

This implies that

$$2\int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x+\sin x}dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x+\sin x}dx + \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x+\cos x}dx$$

$$= \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$$

So $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x + \sin x} dx = \frac{\pi}{4}$.

b) Evaluate the definite integral $\displaystyle\int_0^{\frac{\pi}{2}} \frac{(\sin x)^n}{(\cos x)^n + (\sin x)^n} dx$ for n a positive integer.

Same as the previous problem.

V. Evaluate $\displaystyle\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ using the substitution $u = \pi - x$ and the identities $\sin(\pi - x) = \sin x$ and $\cos(\pi - x) = -\cos x$.

$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = -\int_{\pi}^{0} \frac{(\pi - u)\sin(\pi - u)}{1+\cos^2(\pi - u)} du$$

.

$$= \int_0^{\pi} \frac{(\pi - u)\sin u}{1+\cos^2 u} du$$

So we conclude that

$$2\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

$$= \pi \int_1^{-1} \frac{-1}{1+u^2} du = \pi \int_{-1}^{1} \frac{1}{1+u^2} du = \pi \tan^{-1} u \Big|_{-1}^{1} = \frac{\pi^2}{2}$$

So $\displaystyle\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = \frac{\pi^2}{4}$.

VI. Show that if f is continuous then $\int_0^{\pi} xf(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$ by showing that $\int_0^{\pi}\left(x-\frac{\pi}{2}\right)f(\sin x)\,dx = 0$ using the substitution $u = x-\frac{\pi}{2}$, $\sin\left(x+\frac{\pi}{2}\right)=\cos x$, and symmetry.

$$\int_0^{\pi}\left(x-\frac{\pi}{2}\right)f(\sin x)\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} uf\left(\sin\left(u+\tfrac{\pi}{2}\right)\right)du$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{u}_{ODD}\ \underbrace{f(\cos u)}_{EVEN}\,du$$

So we can conclude that the value of the integral is zero.

Limit Problems

I. What happens if you try L'Hopital's Rule on $\lim\limits_{x\to\infty}\frac{x\sin x}{x^2+1}$?

$$\frac{(x\sin x)'}{(x^2+1)'} = \frac{x\cos x+\sin x}{2x}$$

$$\frac{(x\cos x+\sin x)'}{(2x)'} = \frac{-x\sin x+2\cos x}{2}$$

$\lim\limits_{x\to\infty}\frac{-x\sin x+2\cos x}{2}$ doesn't exist, so L'Hopital doesn't apply.

II. $\lim\limits_{x\to\infty}\frac{x+\sin x}{x}$.

$$\frac{(x+\sin x)'}{(x)'} = \frac{1+\cos x}{1}$$

Again, $\lim_{x\to\infty}\frac{1+\cos x}{1}$ doesn't exist, so L'Hopital doesn't apply, but we can use the double inequality

$$\frac{x-1}{x}\le\frac{x+\sin x}{x}\le\frac{x+1}{x}.$$

III. Find the value of c so that $\lim_{x\to\infty}\left(\frac{x+c}{x-c}\right)^x=9$.

$$\ln\left[\left(\frac{x+c}{x-c}\right)^x\right]=x\ln\left(\frac{x+c}{x-c}\right)=\frac{\ln\left(\frac{x+c}{x-c}\right)}{\frac{1}{x}}$$

$$\lim_{x\to\infty}\frac{\ln\left(\frac{x+c}{x-c}\right)}{\frac{1}{x}}\ \textit{has the form}\ \left(\frac{0}{0}\right),$$

So
$$\frac{\left[\ln\left(\frac{x+c}{x-c}\right)\right]'}{\left(\frac{1}{x}\right)'}=\frac{\frac{x-c}{x+c}\cdot\frac{x-c-(x+c)}{(x-c)^2}}{\frac{-1}{x^2}}$$

$$=\frac{2cx^2}{(x+c)(x-c)}$$

$$\lim_{x\to\infty}\frac{2cx^2}{(x+c)(x-c)}=2c$$

So $\lim_{x\to\infty}\left(\frac{x+c}{x-c}\right)^x=e^{2c}$, and hence that $e^{2c}=9\Rightarrow c=\ln 3$.

IV. Find a simple formula for $\lim_{x\to b}\frac{x^b-b^x}{x^x-b^b}$, for $b>0$.

$$\frac{\left(x^b-b^x\right)'}{\left(x^x-b^b\right)'}=\frac{bx^{b-1}-b^x\ln b}{x^x(1+\ln x)}$$

$$\lim_{x\to b}\frac{bx^{b-1}-b^x\ln b}{x^x(1+\ln x)}=\frac{b^b-b^b\ln b}{b^b(1+\ln b)}=\frac{1-\ln b}{1+\ln b};\left(b\ne e^{-1}\right)$$

V. Find $\lim_{x\to 0}\frac{x^2\sin\left(\frac{1}{x}\right)}{\tan x}$. L'Hopital's Rule won't work, so try something else.

$$\frac{\left[x^2\sin\left(\frac{1}{x}\right)\right]'}{(\tan x)'}=\frac{-\cos\left(\frac{1}{x}\right)+2x\sin\left(\frac{1}{x}\right)}{\sec^2 x}$$

But $\lim_{x\to 0}\frac{-\cos\left(\frac{1}{x}\right)+2x\sin\left(\frac{1}{x}\right)}{\sec^2 x}$ doesn't exist, so L'Hopital doesn't apply.

$$\frac{x^2\sin\left(\frac{1}{x}\right)}{\tan x}=\frac{x}{\tan x}\cdot\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

$$\lim_{x\to 0}\frac{x}{\tan x}=1 \text{ and } \lim_{x\to 0^{\pm}}\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}=\lim_{t\to\pm\infty}\frac{\sin t}{t}=0,$$

So the limit is 0.

VI. Find the following limits:

a) $\lim_{x\to 0}\frac{\ln\left(\frac{e^x-1}{x}\right)}{x}$

First let's verify that L'Hopital's rule applies: $\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{x\to 0}\frac{e^x}{1}=1$, so $\lim_{x\to 0}\frac{\ln\left(\frac{e^x-1}{x}\right)}{x}$ has the $\left(\frac{0}{0}\right)$ form.

$$\frac{\left[\ln\left(\frac{e^x-1}{x}\right)\right]'}{(x)'}=\frac{\frac{x}{e^x-1}\cdot\frac{xe^x-e^x+1}{x^2}}{1}=\frac{xe^x-e^x+1}{xe^x-x}\left(\frac{0}{0}\right)$$

$$\frac{\left(xe^x - e^x + 1\right)'}{\left(xe^x - x\right)'} = \frac{xe^x}{xe^x + e^x - 1}\left(\frac{0}{0}\right)$$

$$\frac{\left(xe^x\right)'}{\left(xe^x + e^x - 1\right)'} = \frac{xe^x + e^x}{xe^x + 2e^x}$$

$$\lim_{x \to 0} \frac{xe^x + e^x}{xe^x + 2e^x} = \frac{1}{2}$$

b) $\lim_{x \to \infty} \frac{\ln\left(\frac{e^x - 1}{x}\right)}{x}$

$$\frac{\left[\ln\left(\frac{e^x - 1}{x}\right)\right]'}{(x)'} = \frac{\frac{x}{e^x - 1} \cdot \frac{xe^x - e^x + 1}{x^2}}{1} = \frac{xe^x - e^x + 1}{xe^x - x}$$

$$= \frac{1 - \frac{1}{x} + \frac{1}{xe^x}}{1 - \frac{1}{e^x}} \to 1$$

VII. Find $\lim_{n \to \infty} \frac{\sqrt[n]{(n+1)(n+2)\cdots(n+n)}}{n}$ by observing the following:

$$\ln\left[\frac{\sqrt[n]{(n+1)(n+2)\cdots(n+n)}}{n}\right] = \frac{1}{n}\left[\ln(n+1) + \ln(n+2) + \cdots + \ln(n+n)\right] - \ln n$$

$$= \frac{1}{n}\left[\ln\left(n\left(1+\tfrac{1}{n}\right)\right) + \ln\left(n\left(1+\tfrac{2}{n}\right)\right) + \cdots + \ln\left(n\left(1+\tfrac{n}{n}\right)\right)\right] - \ln n$$

$$= \frac{1}{n}\left[\ln\left(1+\tfrac{1}{n}\right) + \ln\left(1+\tfrac{2}{n}\right) + \cdots + \ln\left(1+\tfrac{n}{n}\right)\right] + \frac{1}{n}\left[\underbrace{\ln n + \ln n + \cdots \ln n}_{\text{n terms}}\right] - \ln n$$

$$= \frac{1}{n}\left[\ln\left(1+\tfrac{1}{n}\right) + \ln\left(1+\tfrac{2}{n}\right) + \cdots + \ln\left(1+\tfrac{n}{n}\right)\right]$$

The last expression is a Riemann sum of some definite integral.

$$\int_0^1 \ln(1+x)\,dx = \ln 4 - 1$$

VIII. The alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by the Alternating Series Test, but what does it converge to?

Find $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)$, and you'll know the sum of the series.

Method 1: Calculate $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right]$ by rewriting it as $\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{n}{n}}\right]$ and identifying it as a definite integral.

$$\int_0^1 \frac{1}{1+x}\,dx = \ln 2$$

Method 2:

$$\int_{n+1}^{2n+1} \frac{1}{x}\,dx < \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n} < \int_{n}^{2n} \frac{1}{x}\,dx$$

$$\ln\left(\frac{2n+1}{n+1}\right) < \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n} < \ln 2$$

IX. Telescopers

a) $$\sum_{n=1}^{\infty}\left(n^{\frac{1}{n}}-(n+1)^{\frac{1}{n+1}}\right) = \lim_{N\to\infty}\left[\left(1-2^{\frac{1}{2}}\right)+\left(2^{\frac{1}{2}}-3^{\frac{1}{3}}\right)+\cdots+\left(N^{\frac{1}{N}}-(N+1)^{\frac{1}{N+1}}\right)\right]$$

$$= \lim_{N\to\infty}\left(1-(N+1)^{\frac{1}{N+1}}\right) = 0$$

$$\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$$

b) $$= \lim_{N\to\infty}\left[\left(1-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right)+\cdots+\left(\frac{1}{\sqrt{N}}-\frac{1}{\sqrt{N+1}}\right)\right]$$

$$= \lim_{N\to\infty}\left(1-\frac{1}{\sqrt{N+1}}\right) = 1$$

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{n^2+n+1}\right) = \sum_{n=1}^{\infty}\left(\tan^{-1}(n+1) - \tan^{-1}(n)\right)$$

c)

$$= \lim_{N\to\infty}\left[\left(\tan^{-1}2 - \tan^{-1}1\right) + \cdots + \left(\tan^{-1}(N+1) - \tan^{-1}(N)\right)\right]$$

$$= \lim_{N\to\infty}\left(\tan^{-1}(N+1) - \tan^{-1}1\right) = \frac{\pi}{4}$$

Assorted Series

I. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$

For $n > e^{e^2}$, $(\ln n)^{\ln n} > \left(e^2\right)^{\ln n}$, so $\dfrac{1}{(\ln n)^{\ln n}} < \dfrac{1}{\left(e^2\right)^{\ln n}} = \dfrac{1}{n^2}$, so we have convergence by comparison.

II. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(\ln(\ln n))^{\ln n}}$

For $n > e^{e^{e^2}}$, $(\ln(\ln n))^{\ln n} > \left(e^2\right)^{\ln n}$, so $\dfrac{1}{(\ln(\ln n))^{\ln n}} < \dfrac{1}{\left(e^2\right)^{\ln n}} = \dfrac{1}{n^2}$, so we have convergence by comparison.

III. a) Show that $\left(1+\dfrac{1}{n}\right)^{n+1} - \left(1+\dfrac{1}{n}\right)^{n} = \dfrac{\left(1+\frac{1}{n}\right)^{n}}{n}$.

$$\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)^{n} = \left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}-1\right)$$

b) Show that if $\{a_n\}$ is a sequence of positive numbers, then if $\{\ln(a_n)\}$ is decreasing, then $\{a_n\}$ is decreasing. In other words, show that if $\ln(a_{n+1}) \le \ln(a_n)$, then $a_{n+1} \le a_n$.

Suppose that $\ln(a_{n+1}) \le \ln(a_n)$. Since the natural exponential function is monotone, $e^{\ln(a_{n+1})} \le e^{\ln(a_n)} \Rightarrow a_{n+1} \le a_n$

c) For $x>0$, show that $\ln(1+x)\le x$.

$$\text{Since } \ln(1+x)=\int_0^x \frac{1}{1+t}dt \text{ and } \frac{1}{1+t}<1, \text{ we get that } \ln(1+x)<\int_0^x dt=x$$

d) Show that $a_n=\ln\left(\frac{\left(1+\frac{1}{n}\right)^n}{n}\right)$ is a decreasing sequence by showing that $f(x)=x\ln\left(1+\frac{1}{x}\right)-\ln x$ has a negative derivative.

$$\left[x\ln\left(1+\frac{1}{x}\right)-\ln x\right]'=\ln\left(1+\frac{1}{x}\right)-\frac{1}{x+1}-\frac{1}{x}$$

$$=\underbrace{\left[\ln\left(1+\frac{1}{x}\right)-\frac{1}{x}\right]}_{<0}-\underbrace{\frac{1}{x+1}}_{<0}<0$$

e) Determine whether the alternating series $\sum_{n=1}^{\infty}(-1)^n\left[\left(1+\frac{1}{n}\right)^{n+1}-\left(1+\frac{1}{n}\right)^n\right]$ is absolutely convergent, conditionally convergent, or divergent using the previous results.

It is convergent by the Alternating Series Test. The series of absolute values is $\sum_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^n}{n}$, but $\frac{\left(1+\frac{1}{n}\right)^n}{n}>\frac{1}{n}$, so it's not absolutely convergent. Therefore the series is conditionally convergent.

IV. a) Starting with $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$, you get that $xe^x=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}$. Now integrate from $x=0$ to $x=1$ and get $\int_0^1 xe^x dx=\sum_{n=0}^{\infty}\frac{\int_0^1 x^{n+1}dx}{n!}$. Evaluate the integrals on both sides of the equation and find the sum of a series.

$$\sum_{n=0}^{\infty}\frac{1}{(n+2)n!}=1$$

b) You can verity the sum you found in part a) by noticing that $\sum_{n=0}^{\infty}\frac{1}{(n+2)n!}=\sum_{n=0}^{\infty}\frac{(n+1)}{(n+2)!}=\sum_{n=0}^{\infty}\frac{(n+2)-1}{(n+2)!}=\sum_{n=0}^{\infty}\left(\frac{1}{(n+1)!}-\frac{1}{(n+2)!}\right)$. So find the sum of this telescopic series and verify the previous result.

$$\sum_{n=0}^{\infty}\left(\frac{1}{(n+1)!}-\frac{1}{(n+2)!}\right)=\lim_{N\to\infty}\left[\left(1-\frac{1}{2!}\right)+\left(\frac{1}{2!}-\frac{1}{3!}\right)+\cdots+\left(\frac{1}{(N+1)!}-\frac{1}{(N+2)!}\right)\right]$$

$$=\lim_{N\to\infty}\left(1-\frac{1}{(N+2)!}\right)=1$$