Differential Equations

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The AP Mathematics Course Description avoids listing "Differential Equations" as a separate topic, but the subject must be covered in both the AB and BC courses. Presumably, the Test Development Committee avoids such a listing because the topic "Differential Equations" calls to mind *methods* of finding solutions—integrating factors, variation of parameters, etc.—which is out of keeping with the spirit of the courses. Thus both the AB and the BC course descriptions include the following bullets:

- Equations involving derivatives. Verbal descriptions are translated into equations involving derivatives and vice versa.
- Solving separable differential equations and using them in modeling (including the study of the equation $y' = ky$ and exponential growth.)
- Geometric interpretation of differential equations via slope fields and the relationship between slope fields and solution curves for differential equations.

In addition, the BC Course Description includes the following:

+ Numerical solution of differential equations using Euler's method.

+ Solving logistic differential equations and using them in modeling.

Although the term "differential equation" is probably not used at the time, it nevertheless is true that early in the course, when a student is given an acceleration function and asked for the velocity function, or the velocity function and asked for the position function, that student is solving a differential equation. Furthermore, students can learn modelling "rates of change" and begin to use the language that is necessary later for setting up differential equations when they are doing "related rate" type problems. I think we do our students a disservice when we treat related rate problems in a stand alone section. I believe that related rate problems are very beneficial when sprinkled generously throughout the course, both for drill in using the chain rule and for repeated use of the language of rates of change in physical applications.

Let's turn now to the topic of differential equations.

Separable Differential Equations

A separable differential equation is one in which $\frac{dy}{dx}$ can be written as the product of a function of x with a function of y; that is, one of the form

$$\frac{dy}{dx} = g(y) \cdot f(x)$$

With a bow toward Leibniz, we can *separate variables*, and rewrite this equation as

$$\frac{1}{g(y)}\,dy = f(x)dx.$$

To make our own computations simpler, let's write $h(y) = \frac{1}{g(y)}$, so we are now looking at

$$h(y)dy = f(x)dx$$

We integrate both sides getting

$$\int h(y)\,dy = \int f(x)\,dx.$$

Next, letting H(y) and F(x) be the appropriate antiderivatives, we have

$$H(y) = F(x) + C.$$

Now it may seem we have done something strange in this derivation: we antidifferentiated one side of the equation with respect to y, and the other side with respect to x. This appearance is misleading. Actually, we antidifferentiated both sides with respect to x ; what was left unsaid was the assumption that y is a function of x; the chain rule has been used but not cited. The wonderful thing about solving equations is that you can check your answers. Let's return to

$$H(y) = F(x) + C$$

and differentiate both sides with respect to x.

Remembering the chain rule, we obtain

$$H'(y)\, dy/dx = F'(x), \text{ or}$$

$$h(y)\frac{dy}{dx} = f(x)$$

$$\frac{dy}{dx} = f(x)/h(y) = f(x)\cdot g(y)$$

which is what we wanted.

Example #1 Find solutions to the equation $\frac{dy}{dx} = y/x$

Solution: Rewriting, we have

$$\frac{1}{y}\, dy = \frac{1}{x}\, dx$$

$$\int \frac{1}{y}\, dy = \int \frac{1}{x}\, dx$$

$$\ln|y| = \ln|x| + C$$

$$|y| = e^{C}\, |x|$$

We certainly have a surplus of absolute value signs, which is usually handled by first writing $y = \pm\, e^{c}\, x$

or $y = kx,\ k \neq 0.$

Finally observe that $y = 0$, the *constant function* (not a number) is also a solution. So solutions are of the form $y = kx$.

It appears from what we've done that the family of solutions are straight lines through the origin. But now we have to think a little bit. No solution curve can go through a point where $x = 0$. Thus the vertical line that is the x-axis is not a solution curve, which we knew since we are assuming y is a *function* of x. But we also get into an area that has caused some discussion lately. The two pieces A: $y = 2x,\ x > 0$ and B: $y = 2x\ ,\ x < 0$ are in fact two different solution functions that depend on an initial condition.

Question for discussion: what are we saying when we write $\int \frac{1}{x} dx = \ln|x| + C$?

Note: the family of solutions in Example #1 is orthogonal to the family of solutions to the equation $\frac{dy}{dx}$ = -x/y. A solution here is a half-circle, either in the half-plane y > 0, or the lower half-plane with center at the origin. Note that the product of the slope functions is (y/x)(-x/y) = -1 when xy is not 0.

Example #2 We solve $\frac{dy}{dx} = \frac{-x}{y}$ more formally, showing the chain rule.

Solution: Rewrite the equation as $y\frac{dy}{dx} = -x,\ y \neq 0$. Remembering y is a function of x, now integrate both sides with respect to x

$$\int y \frac{dy}{dx} dx = \int -x \, dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C,$$

or, $$x^2 + y^2 = C, \text{ with } y \neq 0.$$

Note, if an initial condition had been given, such as $y(3) = -4$, that would give us C= 25 and the solution would be

$$y = -\sqrt{25 - x^2}\,,\ -5 < x < 5$$

In Example #1, note the important use absolute value when integrating the reciprocal function; if we omitted the absolute value we would only have found solutions with positive slope, missing "half" the solutions. The inclusion of absolute values is also vitally important when going through the steps to solve the equation $\frac{dy}{dx}$ = ky, but having just done a similar example, I will not repeat them here, but I do go through the steps very carefully in class. Having done that however, I want my students to *recognize* in later work that the solution to $\frac{dA}{dt} = kA$ is $A = A_0e^{kt}$ *without* separating variables. This recognition then lets us handle the wide variety of "Newton's-Law-of-Cooling" problems as follows:

Example #3: (AB 6 1993) (Similar to AB 5 2011) A population of wolves is increasing at a rate directly proportional to $800 - P(t)$, where the constant of proportionality is k.

(a) If $P(0)= 500$, find $P(t)$ in terms of t and k.

(b) If $P(2)=700$, find k.

(c) Find $\lim_{t\to\infty} P(t)$

Solution #1: Separate variables

(a) The equation is $\frac{dP(t)}{dt} = k(800 - P(t))$, or

$$\frac{1}{800 - P(t)}\frac{dP(t)}{dt} = k$$

Integrate with respect to t

$$\int \frac{1}{800 - P(t)} P'(t)dt = \int k\, dt$$

$$-\ln|800 - P(t)| = kt + C$$

$$\ln|800 - P(t)| = -kt + C$$

$$|800 - P(t)| = e^{-kt+C} = e^C e^{-kt} = C_1 e^{-kt}, \quad C_1 > 0$$

$$800 - P(t) = C_2 e^{-kt}, C_2 \neq 0$$

Use $P(0)= 500$ to write $800 - 500 = C_2 e^{-k0} = C_2$,

So $P(t) = 800 - 300e^{-kt}$

(b) $700 = 800 - e^{-2k}$, or $k = \frac{\ln 3}{2}$.

(c) $P(t) = 800 - 300e^{-\frac{t\ln 3}{2}} = 800 - 300(\frac{1}{3})^{\frac{t}{2}}$, so $\lim_{t\to\infty} P(t)$=800.

Solution #2. Use a substitution.

Start with $\frac{dP}{dt} = k(800 - P)$

Use a substitution y = 800 - P. Differentiating this equation with respect to t, we get

$$(*)\ \frac{dy}{dt} = -\frac{dp}{dt}.$$

Now the substitution was chosen so that $\frac{dp}{dt}$ = ky, thus

$$(**)\ \frac{dy}{dt} = -ky.$$

Can we now recognize the exponential differential equation, so $y = y_0e^{-kt}$.

Now y = 800 - P, so we have

$800 - P = y_0\ e^{-kt}$ or

$$P = 800 - y_0e^{-kt}.$$

We can now use the first initial condition P = 500 when t = 0 to see $y_0 = 300$, and so the solution is

$$P = 800 - 300\ e^{-kt}.$$

The rest of the problem follows as in solution 1. **End of Solution**

Remark: The point of second solution in Example 3 was to demonstrate that this type of problem is simply a translation of the more familiar exponential growth or decay problem.

Example #4. Solve $\frac{dy}{dx}$ = 600 - 2y if y = 100 when x = 0.

Solution: Let w = 600 - 2y so

$\frac{dw}{dx} = -2\frac{dy}{dx}$. Our substitution was chosen so that w = $\frac{dy}{dx}$, so our new equation , not involving y, is $\frac{dw}{dx} = -2w$. The solution is $w = w_0e^{-2x}$. Note $w_0 \neq y_0$, in fact $w_0 = 600 - 2y_0 = 600 - 200 = 400$. Thus $w = 400\ e^{-2x}$, or $600 - 2y = 400\ e^{-2x}$.

Finally, $2y = 600 - 400\ e^{-2x}$, or

$y = 300 - 200\ e^{-2x}$. **End of Solution**

Here is a problem that calls for thinking about setting up a differential equation, and analyzing the behavior it is modeling without actually solving the equation.

Example #5. Water flows into a cylindrical tank at the rate of 40 cubic feet per minute. The tank is 20 feet tall and the base of the tank has diameter 6 feet. Water flows out of the tank at a rate proportional to the square root of the depth of the water. Let V be the volume of water in the tank at time t, and let h be the corresponding depth of the water.

(a) Write a differential equation for the rate of change of the volume of water in the tank using only the variables V and t.

(b) If water flows out at a rate of 20 cubic feet per minute when the depth is four feet, write an equation for $\frac{dV}{dt}$ that does not involve any unspecified constant.

(c) If at time t = 0, the tank is empty and the flow begins. Does the tank ever fill up? Explain.

Solution:

(a) $\frac{dV}{dt}$ = flow in - flow out.

flow in = 40; flow out = $K\sqrt{h}$, K a constant, and since $V = 9\pi h$, $h = \frac{V}{9\pi}$.

$$\text{Thus } \frac{dV}{dt} = 40 - K\sqrt{\frac{V}{9\pi}}.$$

(b) When h = 4, flow out is 20, so

$20 = K\sqrt{4}$; k = 10

$$\text{Thus } \frac{dV}{dt} = 40 - 10\sqrt{\frac{V}{9\pi}}.$$

(c) The volume of water in the tank is increasing when

$$40 - 10\sqrt{h} > 0$$

or $4 > \sqrt{h}$.

Thus the water level only increases when $h < 16$.

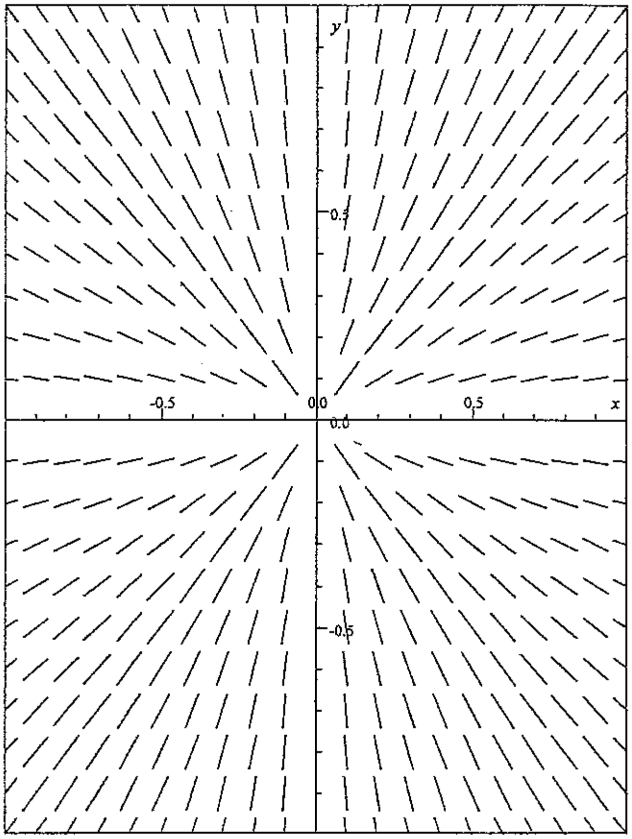
Answer: No, the tank never is full; the depth of the water approaches 16 feet, but is never greater than 16 feet. **End of Solution**

Example #6 Sometimes separating is not trivial. Solve $y' = xy^2 + x$

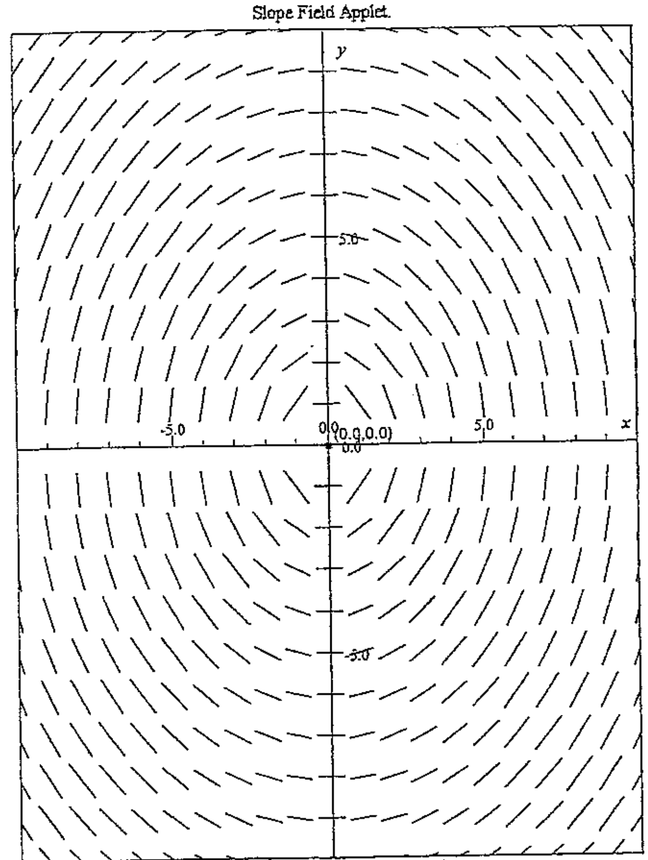
Solution: The first step is perhaps to change notation, and write $\frac{dy}{dx} = xy^2 + x$. Don't give in to the temptation to subtract! Write $xy^2 + x$ as $x(y^2 + 1)$, so $\frac{1}{y^2 + 1} dy = xdx$,

or $\arctan(y) = \frac{x^2}{2} + C$. Solving for y, we have $y = \tan(\frac{x^2}{2} + C)$. Note the position of the C. Thus with an initial condition $y(0) = 1$, we have $y = \tan(\frac{x^2}{2} + \frac{\pi}{4})$.

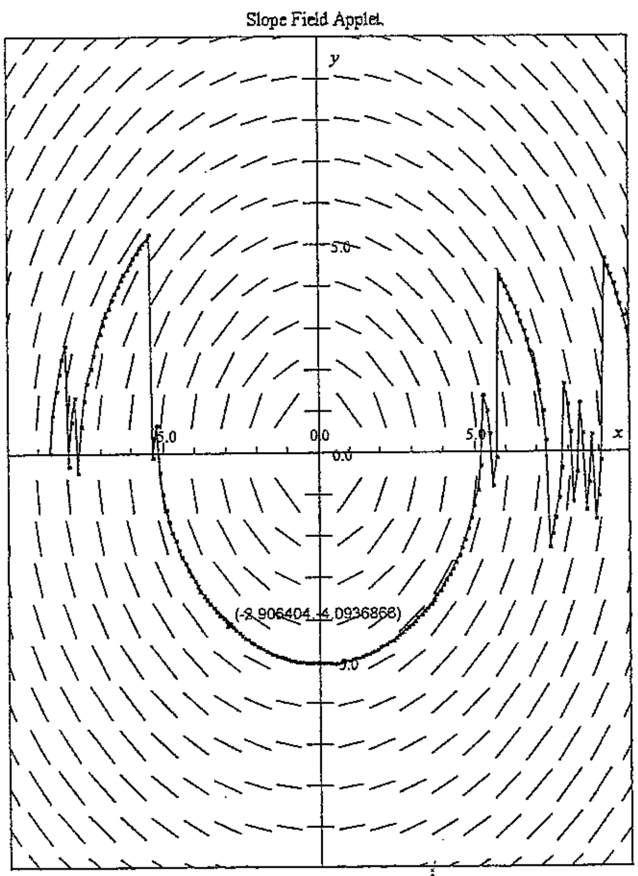
Let's look at some slope field pictures of the equations we have been talking about.



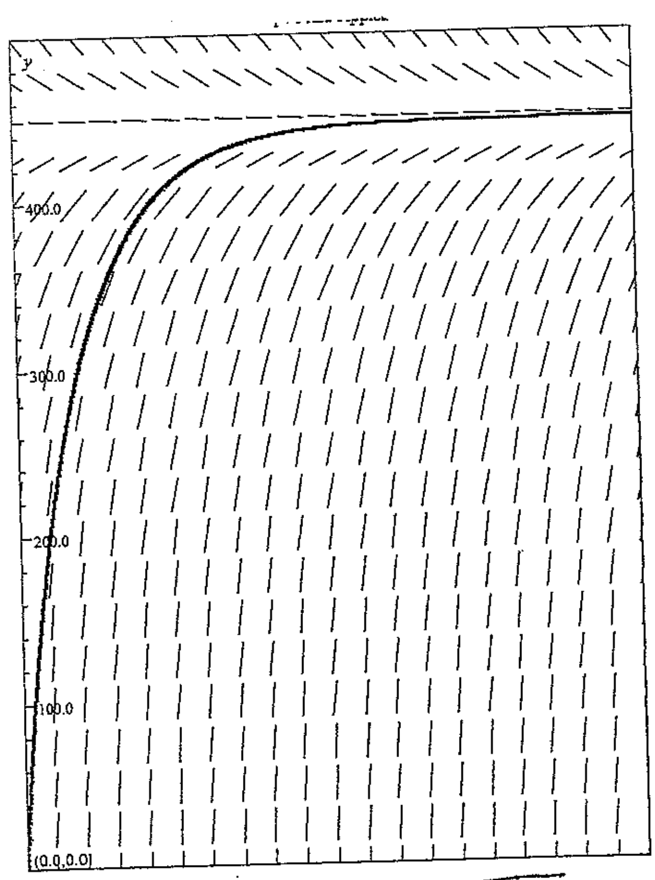
$y' = y/x$



$y' = -\frac{x}{y}$



A "Solution" to $y' = -\frac{x}{y}$



$\frac{dV}{dt} = 40 - 10\sqrt{\frac{V}{9\pi}}$