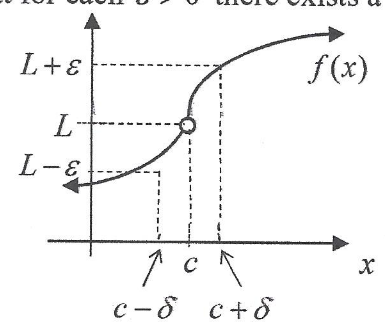
Ⓘ

Formal Definition of a Limit

Formal Definition of a Limit: (Epsilon-Delta)

Let f be a function defined on an open interval containing c (except possibly **at** c) and let L be a real number. The statement $\lim_{x\to c} f(x) = L$ means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x-c| < \delta$, then $|f(x) - L| < \varepsilon$.



Since $|x-c|$ is the distance from x to c and $|f(x)-L|$ is the distance from $f(x)$ to L and since ε can be arbitrarily small, the definition of a limit can be expressed in words:

$\lim_{x\to c} f(x) = L$ means that the values of $f(x)$ can be made as close as we please to L by taking x close enough to c (but not equal to c).

More formally:

$\lim_{x\to c} f(x) = L$ means that for every $\varepsilon > 0$ (no matter how small ε is), we can find $\delta > 0$ such that if x lies in the open interval $(c-\delta,\ c+\delta)$ and $x \neq c$, then $f(x)$ lies in the open interval $(L-\varepsilon,\ L+\varepsilon)$.

EXAMPLE 1: If $\lim_{x\to 2}(3x+4) = 10$, find a number $\delta > 0$ such that $|f(x) - L| < 0.01$ when $|x-c| < \delta$.

$$|(3x+4)-10| < 0.01$$
$$|3x+4-10| < 0.01$$
$$|3x-6| < 0.01$$
$$|3(x-2)| < 0.01$$
$$3|x-2| < 0.01$$
$$|x-2| < 0.03$$

* *Conclusion: If x is within distance 0.03 of 2, then $3x+4$ will be within distance 0.01 of 10.*

EXAMPLE 2: Use the $\varepsilon - \delta$ definition of limit to show that $\lim_{x\to 2}(3x+4) = 10$

- *You must show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|(3x+4)-10| < \varepsilon$ when $0 < |x-2| < \delta$.*
- *Your choice of δ depends on ε, so establish a connection between $|(3x+4)-10|$ and $|x-2|$.*

From the previous example, $|(3x+4)-10| = |3x-6| = 3|x-2| < \varepsilon$. For a given $\varepsilon > 0$, choose $\delta = \dfrac{\varepsilon}{3}$.

Hence, $0 < |x-2| < \delta = \dfrac{\varepsilon}{3}$ implies that $|(3x+4)-10| = 3|x-2| < 3\left(\dfrac{\varepsilon}{3}\right) = \varepsilon$

(I)

EXAMPLE 3: Use the $\varepsilon-\delta$ definition of limit to prove that $\lim_{x\to 2} x^2 = 4$

You must show that for each $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $|x^2-4| < \varepsilon$ *when* $0<|x-2|<\delta$

$$|x^2-4| = |x-2||x+2| \quad < \varepsilon \quad \text{if } |x-2|<\delta$$

For all x in the interval $(1, 3)$, $|x+2|<5$.

Let δ be the $\min\left\{\frac{\varepsilon}{5}, 1\right\}$.

So whenever $0<|x-2|<\delta$, we have:

$$|x^2-4| = |x-2||x+2| < \left(\frac{\varepsilon}{5}\right)(5) = \varepsilon$$

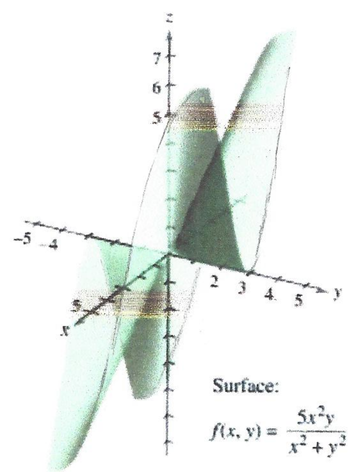
We want to bound the factor of $|x-2|$
- If $\delta \le 1$, then $-1 < x-2 < 1$ so $3 < x+2 < 5$ so $|x+2| < 5$

and $|x-2||x+2| < 5\delta$

WHY DO WE CARE ABOUT THIS?? Let's consider functions of several variables.

Evaluate $\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2}$

- *The limits of the numerator and denominator are both 0, so the existence (or nonexistence) of a limit by taking the limits of the numerator and denominator separately and then dividing cannot be determined.*
- *From the graph, it is reasonable to assume that the limit might be* $L = 0$



Note: $|y| \le \sqrt{x^2+y^2}$ and $\frac{x^2}{x^2+y^2} \le 1$

Thus, in a δ-neighborhood about $(0, 0)$, we have $0 < \sqrt{x^2+y^2} < \delta$, it follows that, for $(x, y) \ne (0, 0)$,

$$|f(x,y)-0| = \left|\frac{5x^2y}{x^2+y^2}\right|$$

$$= 5|y|\left(\frac{x^2}{x^2+y^2}\right)$$

$$\le 5|y|$$

$$\le 5\sqrt{x^2+y^2}$$

$$< 5\delta$$

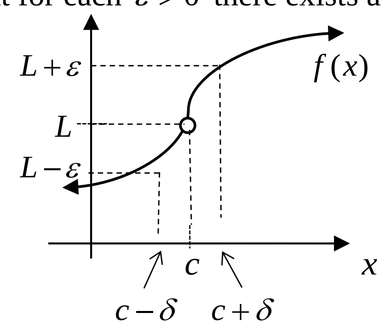
Then, choosing $\delta = \frac{\varepsilon}{5}$, it follows that $|f(x,y)-0| < \varepsilon$.

Hence, $\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2} = 0$

Formal Definition of a Limit

Formal Definition of a Limit: (Epsilon-Delta)

Let f be a function defined on an open interval containing c (except possibly **at** c) and let L be a real number. The statement $\lim_{x \to c} f(x) = L$ means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.



Since $|x - c|$ is the distance from x to c and $|f(x) - L|$ is the distance from $f(x)$ to L and since ε can be arbitrarily small, the definition of a limit can be expressed in words:

$\lim_{x \to c} f(x) = L$ *means that the values of* $f(x)$ *can be made as close as we please to* L *by taking* x *close enough to* c *(but not equal to* c *).*

More formally:

$\lim_{x \to c} f(x) = L$ *means that for every* $\varepsilon > 0$ *(no matter how small* ε *is), we can find* $\delta > 0$ *such that if* x *lies in the open interval* $(c - \delta, c + \delta)$ *and* $x \neq c$, *then* $f(x)$ *lies in the open interval* $(L - \varepsilon, L + \varepsilon)$.

EXAMPLE 1: If $\lim_{x \to 2}(3x + 4) = 10$, find a number $\delta > 0$ such that $|f(x) - L| < 0.01$ when $|x - c| < \delta$.

$$|(3x + 4) - 10| < 0.01$$
$$|3x + 4 - 10| < 0.01$$
$$|3x - 6| < 0.01$$
$$|3(x - 2)| < 0.01$$
$$3|x - 2| < 0.01$$
$$|x - 2| < 0.03$$

Conclusion: If x *is within distance* 0.03 *of* 2, *then* $3x + 4$ *will be within distance* 0.01 *of* 10.

EXAMPLE 2: Use the $\varepsilon - \delta$ definition of limit to show that $\lim_{x \to 2}(3x + 4) = 10$

- *You must show that for each* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $|(3x + 4) - 10| < \varepsilon$ *when* $0 < |x - 2| < \delta$.
- *Your choice of* δ *depends on* ε, *so establish a connection between* $|(3x + 4) - 10|$ *and* $|x - 2|$.

From the previous example, $|(3x + 4) - 10| = |3x - 6| = 3|x - 2| < \varepsilon$. For a given $\varepsilon > 0$, choose $\delta = \dfrac{\varepsilon}{3}$.

Hence, $0 < |x - 2| < \delta = \dfrac{\varepsilon}{3}$ implies that $|(3x + 4) - 10| = 3|x - 2| < 3\left(\dfrac{\varepsilon}{3}\right) = \varepsilon$

EXAMPLE 3: Use the $\varepsilon - \delta$ definition of limit to prove that $\lim_{x \to 2} x^2 = 4$

You must show that for each $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $\left| x^2 - 4 \right| < \varepsilon$ *when* $0 < \left| x - 2 \right| < \delta$

$$\left| x^2 - 4 \right| = |x-2|\,|x+2|$$

For all x in the interval $(1, 3)$, $|x+2| < 5$.

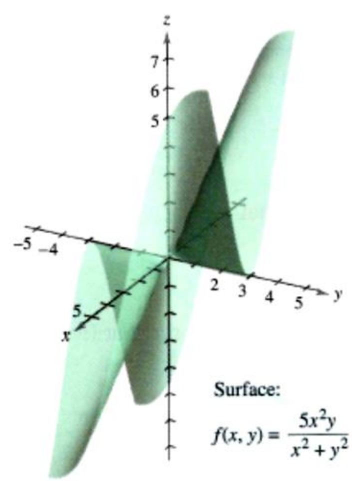
Let δ be the $\min\left\{\frac{\varepsilon}{5}, 1\right\}$.

So whenever $0 < |x-2| < \delta$, we have:

$$\left| x^2 - 4 \right| = |x-2|\,|x+2| < \left(\frac{\varepsilon}{5}\right)(5) = \varepsilon$$

WHY DO WE CARE ABOUT THIS?? Let's consider functions of several variables.

Evaluate $\lim_{(x,y)\to(0,0)} \frac{5x^2 y}{x^2 + y^2}$



- *The limits of the numerator and denominator are both 0, so the existence (or nonexistence) of a limit by taking the limits of the numerator and denominator separately and then dividing cannot be determined.*

- *From the graph, it is reasonable to assume that the limit might be* $L = 0$

Note: $|y| \le \sqrt{x^2 + y^2}$ and $\frac{x^2}{x^2 + y^2} \le 1$

Thus, in a δ-neighborhood about $(0, 0)$, we have $0 < \sqrt{x^2 + y^2} < \delta$, it follows that, for $(x, y) \neq (0, 0)$,

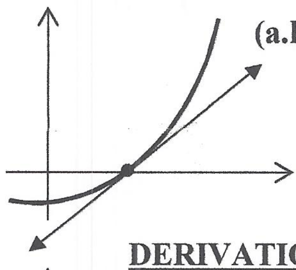
$$\begin{aligned} \left| f(x,y) - 0 \right| &= \left| \frac{5x^2 y}{x^2 + y^2} \right| \\ &= 5|y|\left(\frac{x^2}{x^2 + y^2}\right) \\ &\le 5|y| \\ &\le 5\sqrt{x^2 + y^2} \\ &< 5\delta \end{aligned}$$

Then, choosing $\delta = \frac{\varepsilon}{5}$, it follows that $\left| f(x,y) - 0 \right| < \varepsilon$.

Hence, $\lim_{(x,y)\to(0,0)} \frac{5x^2 y}{x^2 + y^2} = 0$

II

NEWTON'S METHOD

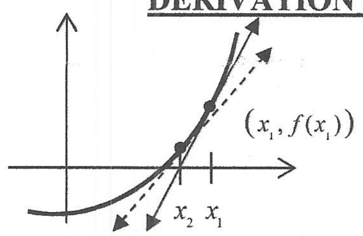


(a.k.a. *"also known as"* Newton-Raphson Method)

Newton's Method involves a recursive formula which approximates the root(s) of continuous functions. (*"Recursive" – the answer you get is then plugged into the formula repetitively.*)

{ IVT
convergence

DERIVATION OF THE FORMULA FOR NEWTON'S METHOD:



The **slope** is given by $f'(x)$

The **tangent** touches the curve at the point $(x_1, f(x_1))$

The **tangent line equation** in Point-Slope Form is:

$$y - f(x_1) = f'(x_1)(x - x_1) \qquad [\textit{from } y - y_1 = m(x - x_1)]$$

Since the tangent line intersects the $x-$*axis at* $(x_2, 0)$:

$$0 - f(x_1) = f'(x_1)(x_2 - x_1) \qquad \textit{Substitute the } x - \textit{intercept.}$$

$$-f(x_1) = x_2 f'(x_1) - x_1 f'(x_1) \qquad \textit{Distribute } f'(x_1).$$

Begin to isolate x_2 *term.*

$$\frac{x_1 f'(x_1)}{f'(x_1)} - \frac{f(x_1)}{f'(x_1)} = \frac{x_2 f'(x_1)}{f'(x_1)} \qquad \textit{Divide each term by } f'(x_1).$$

$$x_1 - \frac{f(x_1)}{f'(x_1)} = x_2 \qquad \textit{Simplify the equation.}$$

$$\therefore\ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad \textit{This finds the "next" } x.$$

By similar reasoning: $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$, etc.

Newton's Method: $\quad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0$

EXAMPLE 1: Use Newton's Method to find the real, positive root for $f(x)$, aproximated to five decimal places.

$f(x) = x^3 + x^2 + x - 1$ *Use IVT to approximate the location for the root.*

$f(0) = -1$ and $f(1) = 2$, $\therefore \exists$ a root, r, in $(0, 1)$

Let $x_1 = 1$ $\quad \therefore\ x_2 = 1 - \dfrac{f(1)}{f'(1)}$ $\quad$ *Find* $f'(x)$, $f'(1)$, and $f(1)$

$f'(x) = 3x^2 + 2x + 1$, $f'(1) = 6$, $f(1) = 2$ $\qquad x_2 = 1 - \dfrac{2}{6} = 1 - \dfrac{1}{3} = \dfrac{2}{3}$

$x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} \approx 0.55556$ $\qquad x_4 = x_3 - \dfrac{f(x_3)}{f'(x_3)} \approx .054382$

$x_5 = x_4 - \dfrac{f(x_4)}{f'(x_4)} \approx 0.54369$ $\qquad x_6 = x_5 - \dfrac{f(x_5)}{f'(x_5)} \approx 0.54369$

When 2 iterations repeat themselves, you have found the closest approximation for the root.

EXAMPLE 2: Use Newton's Method to find the real, positive root for $f(x)$, aproximated to five decimal places.

$f(x) = x^3 - 7$ $f(1) < 0$, $f(2) > 0$ let $x_0 = 2$

$x_1 = 2 - \frac{f(2)}{f'(2)} \approx 1.91\overline{6}$

$x_2 \approx 1.91\overline{6} - \frac{f(1.91\overline{6})}{f'(1.91\overline{6})} \approx 1.912938458$

$x_3 \approx x_2 - \frac{f(x_2)}{f'(x_2)} \approx 1.912931183 \approx x_4$

Also, $f(x_4) \approx 1 \times 10^{-13}$

NOTE: Newton's Method does not always work.

EXAMPLE 3: Use Newton's Method to find the real, positive root for $f(x)$, aproximated to five decimal places.

$f(x) = x^3 - 3x^2 + x - 1$

What happens if you choose $x_0 = 1$? You get $x_0 = 1,\ x_1 = 0,\ x_2 = 1,\ x_3 = 0$, etc.

IVT is helpful!

$f(2) < 0$

$f(3) > 0$

$x_0 = 3 - \frac{f(3)}{f'(3)} = 2.8$

$x_1 = 2.8 - \frac{f(2.8)}{f'(2.8)} \approx 2.76994818$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 2.76929266$

Answer: $x = 2.769292354$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 2.769292354$

Sometimes, you will have an unusually slow convergence.

EXAMPLE 3: Use Newton's Method to find the real, positive root for $f(x)$, aproximated to five decimal places.

$f(x) = \frac{(x-1)^2}{x^2+1}$ $f(0) = 1$, $f(1) = 0$ } Let's see how long it takes to converge to 1.

$x_0 = 0$

$x_1 = 0 - \frac{f(0)}{f'(0)} = 0.5$

$x_2 = 0.5 - \frac{f(0.5)}{f'(0.5)} \approx 0.7083333$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 0.8365303$

$x_4 \approx 0.912179$

$x_5 \approx 0.95425$

$x_6 \approx 0.976614$

$x_7 \approx 0.9881719$

$x_8 \approx 0.9940512$

$x_9 \approx 0.9970168$

$x_{10} \approx 0.9985062$

$x_{11} \approx 0.9992525$

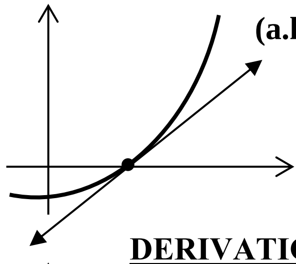
$x_{12} \approx 0.9996261$

Answer: $x = 0.9996261$

$x_{13} \approx 0.999813026$

Comment: Approximations are at the heart of calculus. Here, the tangetn line is thought of as an approximation of a curve and used to approximate solutions of equations for which algebra fails.

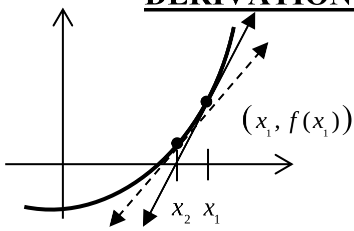
NEWTON'S METHOD



(a.k.a. *"also known as"* Newton-Raphson Method)

Newton's Method involves a <u>recursive</u> formula which approximates the root(s) of continuous functions. (*"Recursive" – the answer you get is then plugged into the formula repetitively.*)

DERIVATION OF THE FORMULA FOR NEWTON'S METHOD:



The **slope** is given by $f'(x)$

The **tangent** touches the curve at the point $(x_1, f(x_1))$

The **tangent line equation** in Point-Slope Form is:

$$y - f(x_1) = f'(x_1)(x - x_1) \qquad [\textit{from } y - y_1 = m(x - x_1)]$$

Since the tangent line intersects the $x-$*axis at* $(x_{2,} 0)$:

$$0 - f(x_1) = f'(x_1)(x_2 - x_1) \qquad \textit{Substitute the } x-\textit{intercept.}$$

$$-f(x_1) = x_2 f'(x_1) - x_1 f'(x_1) \qquad \textit{Distribute } f'(x_1).$$

Begin to isolate x_2 *term.*

$$\frac{x_1 f'(x_1)}{f'(x_1)} - \frac{f(x_1)}{f'(x_1)} = \frac{x_2 f'(x_1)}{f'(x_1)} \qquad \textit{Divide each term by } f'(x_1).$$

$$x_1 - \frac{f(x_1)}{f'(x_1)} = x_2 \qquad \textit{Simplify the equation.}$$

$$\therefore\ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad \textit{This finds the "next" } x.$$

By similar reasoning: $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$, etc.

<u>Newton's Method</u>: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0$

<u>EXAMPLE 1</u>: Use Newton's Method to find the real, positive root for $f(x)$, aproximated to five decimal places.

$$f(x) = x^3 + x^2 + x - 1 \qquad \textit{Use IVT to approximate the location for the root.}$$

$f(0) = -1$ and $f(1) = 2$, $\therefore \exists$ a root, r, in $(0, 1)$

Let $x_1 = 1$ $\qquad \therefore\ x_2 = 1 - \dfrac{f(1)}{f'(1)}$ $\qquad$ *Find* $f'(x)$, $f'(1)$, and $f(1)$

$f'(x) = 3x^2 + 2x + 1$, $f'(1) = 6$, $f(1) = 2$ $\qquad x_2 = 1 - \dfrac{2}{6} = 1 - \dfrac{1}{3} = \dfrac{2}{3}$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 0.55556 \qquad x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx .054382$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} \approx 0.54369 \qquad x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} \approx 0.54369$$

When 2 iterations repeat themselves, you have found the closest approximation for the root.

EXAMPLE 2: Use Newton's Method to find the real, positive root for $f(x)$, aproximated to five decimal places.

$$f(x) = x^3 - 7$$

NOTE: Newton's Method does not always work.

EXAMPLE 3: Use Newton's Method to find the real, positive root for $f(x)$, aproximated to five decimal places.

$$f(x) = x^3 - 3x^2 + x - 1$$

What happens if you choose $x_0 = 1$? You get $x_0 = 1,\ x_1 = 0,\ x_2 = 1,\ x_3 = 0$, etc.

Answer: $x = 2.769292354$

Sometimes, you will have an unusually slow convergence.

EXAMPLE 3: Use Newton's Method to find the real, positive root for $f(x)$, aproximated to five decimal places.

$$f(x) = \frac{(x-1)^2}{x^2+1}$$

Answer: $x = 0.9996261$

Comment: Approximations are at the heart of calculus. Here, the tangetn line is thought of as an approximation of a curve and used to approximate solutions of equations for which algebra fails.

Riemann Sums

TOOLBOX:

$\sum_{k=1}^{n} h \cdot w$ $\quad h = f(x_k) \quad x_k = a + (\Delta x) \cdot k \quad w = \Delta x = \frac{b-a}{n}$

$\sum_{k=1}^{n} f(x_k) \cdot \Delta x$ — **APPROXIMATION**

$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k) \cdot \Delta x = \int_a^b f(x)\,dx$ — **BETTER APPROXIMATION**

EXAMPLE 1: Write $\int_1^5 (x^2 + 3)\,dx$ as a Riemann Sum.

[1,5] a = 1

$\Delta x = \frac{5-1}{n}$ $\quad \boxed{\Delta x = \frac{4}{n}}$

$x_k = a + \Delta x \cdot k$ $\quad \boxed{x_k = 1 + \frac{4}{n} \cdot k}$

$f(x) = x^2 + 3$

$f(x_k) = (1 + \frac{4k}{n})^2 + 3$

$\boxed{\lim_{n \to \infty} \sum_{k=1}^{n} \left[(1 + \frac{4k}{n})^2 + 3\right] \cdot \frac{4}{n}}$

EXAMPLE 2: Write $\lim_{n \to \infty} \sum_{k=1}^{n} \left(\sqrt{1 + \frac{3k}{n}} \cdot \frac{1}{n} \right)$ as a definite integral.

$f(x) = \sqrt{\ldots\ldots}$

$\Delta x = \frac{1}{n}$

so $b - a = 1$

$\frac{3k}{n} \Rightarrow 3(\frac{1}{n} \cdot k)$

$\Rightarrow 3(\Delta x \cdot k)$ so $a = 0$, then $b = 1$

$f(x) = \sqrt{1 + 3x}$

$\boxed{\int_0^1 \sqrt{1 + 3x}\,dx}$

EXAMPLE 3: Which of the limits is equal to $\int_2^5 x^2\,dx$? *(from Big Book AB Test, p. 29 #30)*

(A) $\lim_{n \to \infty} \sum_{k=1}^{n} \left(2 + \frac{k}{n}\right)^2 \frac{1}{n}$

(B) $\lim_{n \to \infty} \sum_{k=1}^{n} \left(2 + \frac{k}{n}\right)^2 \frac{3}{n}$

(C) $\lim_{n \to \infty} \sum_{k=1}^{n} \left(2 + \frac{3k}{n}\right)^2 \frac{1}{n}$

(D) $\lim_{n \to \infty} \sum_{k=1}^{n} \left(2 + \frac{3k}{n}\right)^2 \frac{3}{n}$

Toolbox:

$dx = \Delta x = \frac{5-2}{n} = \frac{3}{n}$

(eliminate A, C)

$a = 2$

$f(x) = x^2$

$f(x_k) = (2 + \frac{3k}{n})^2$

$x_k = a + \Delta x \cdot k$

$x_k = 2 + \frac{3}{n} \cdot k$

$\boxed{\lim_{n \to \infty} \sum_{k=1}^{n} (2 + \frac{3k}{n})^2 \cdot (\frac{3}{n})}$

(D)

EXAMPLE 4: The function f is given by $f(x) = \ln x$. The graph of f is shown at right. Which of the following limits is equal to the area of the shaded region? *(from Big Book BC Test, p. 29 #29)*

(A) $\lim_{n \to \infty} \sum_{k=1}^{n} \left(1 + \ln\left(\frac{3k}{n}\right)\right) \frac{3}{n}$

(B) $\lim_{n \to \infty} \sum_{k=1}^{n} \ln\left(1 + \frac{3k}{n}\right) \frac{3}{n}$

(C) $\lim_{n \to \infty} \sum_{k=1}^{n} \ln\left(\frac{4}{n}\right)\left(1 + \frac{4k}{n}\right)$

(D) $\lim_{n \to \infty} \sum_{k=1}^{n} \ln\left(1 + \frac{4k}{n}\right) \frac{4}{n}$

Toolbox:

$\Delta x = \frac{4-1}{n} = \frac{3}{n}$ $\quad$ [1, 4] $\quad a = 1$

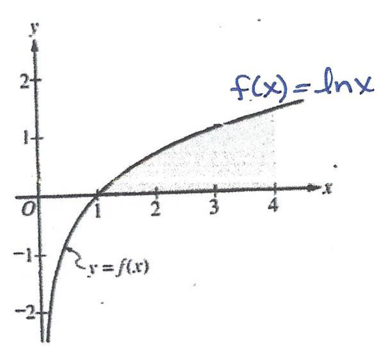
(elim. C, D)

$x_k = 1 + \frac{3k}{n}$

$f(x_k) = \ln(1 + \frac{3k}{n})$

$\boxed{\lim_{n \to \infty} \sum_{k=1}^{n} \ln(1 + \frac{3k}{n}) \cdot \frac{3}{n}}$

(B)



III

16.1 DOUBLE INTEGRALS OVER RECTANGLES

In much the same way that our attempt to solve the area problem led to the definition of a definite integral, we now seek to find the volume of a solid and in the process we arrive at the definition of a double integral.

REVIEW OF THE DEFINITE INTEGRAL

First let's recall the basic facts concerning definite integrals of functions of a single variable. If $f(x)$ is defined for $a \leq x \leq b$, we start by dividing the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/n$ and we choose sample points x_i^* in these subintervals. Then we form the Riemann sum

1 $$\sum_{i=1}^{n} f(x_i^*) \, \Delta x$$

and take the limit of such sums as $n \to \infty$ to obtain the definite integral of f from a to b:

2 $$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \, \Delta x$$

In the special case where $f(x) \geq 0$, the Riemann sum can be interpreted as the sum of the areas of the approximating rectangles in Figure 1, and $\int_a^b f(x) \, dx$ represents the area under the curve $y = f(x)$ from a to b.

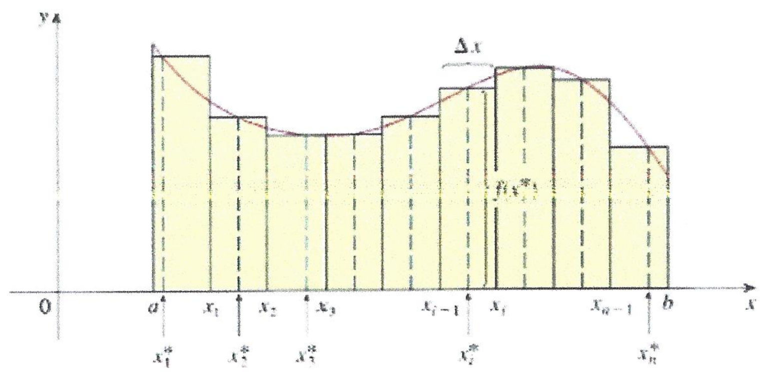


FIGURE 1

VOLUMES AND DOUBLE INTEGRALS

In a similar manner we consider a function f of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \ c \leq y \leq d\}$$

and we first suppose that $f(x, y) \geq 0$. The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f, that is,

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq f(x, y), \ (x, y) \in R\}$$

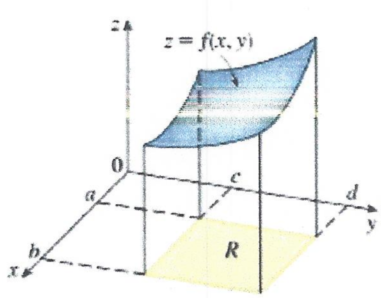


FIGURE 2

(See Figure 2.) Our goal is to find the volume of S.

The first step is to divide the rectangle R into subrectangles. We accomplish this by dividing the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/m$ and dividing $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = (d - c)/n$. By draw-

III

p. 3

ing lines parallel to the coordinate axes through the endpoints of these subintervals, as in Figure 3, we form the subrectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leq x \leq x_i,\ y_{j-1} \leq y \leq y_j\}$$

each with area $\Delta A = \Delta x\, \Delta y$.

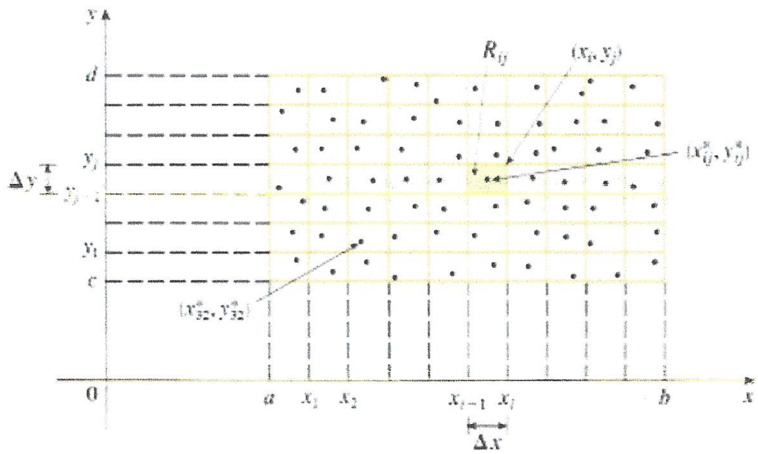


FIGURE 3
Dividing *R* into subrectangles

If we choose a **sample point** (x_{ij}^*, y_{ij}^*) in each R_{ij}, then we can approximate the part of *S* that lies above each R_{ij} by a thin rectangular box (or "column") with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$ as shown in Figure 4. (Compare with Figure 1.) The volume of this box is the height of the box times the area of the base rectangle:

$$f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

If we follow this procedure for all the rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volume of *S*:

3 $$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

(See Figure 5.) This double sum means that for each subrectangle we evaluate *f* at the chosen point and multiply by the area of the subrectangle, and then we add the results.

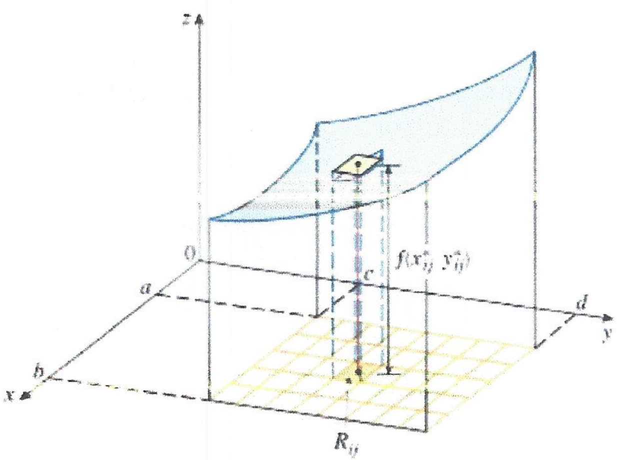


FIGURE 4

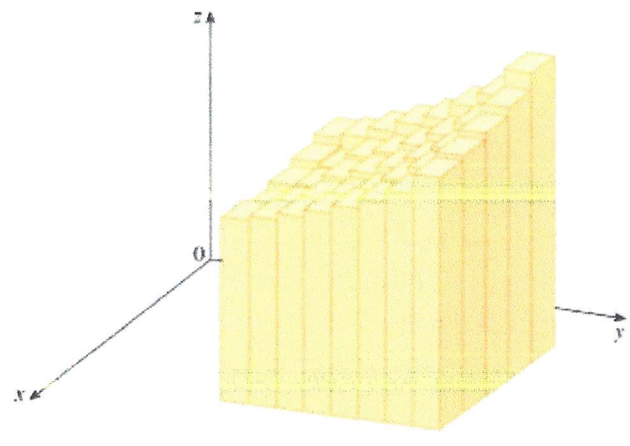


FIGURE 5

III

The sum in Definition 5,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

is called a **double Riemann sum** and is used as an approximation to the value of the double integral. [Notice how similar it is to the Riemann sum in (1) for a function of a single variable.] If f happens to be a *positive* function, then the double Riemann sum represents the sum of volumes of columns, as in Figure 5, and is an approximation to the volume under the graph of f and above the rectangle R.

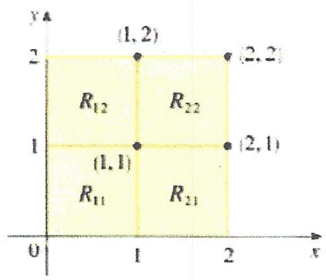


FIGURE 6

EXAMPLE 1 Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.

SOLUTION The squares are shown in Figure 6. The paraboloid is the graph of $f(x, y) = 16 - x^2 - 2y^2$ and the area of each square is 1. Approximating the volume by the Riemann sum with $m = n = 2$, we have

$$\begin{aligned} V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i, y_j)\,\Delta A \\ &= f(1, 1)\,\Delta A + f(1, 2)\,\Delta A + f(2, 1)\,\Delta A + f(2, 2)\,\Delta A \\ &= 13(1) + 7(1) + 10(1) + 4(1) = 34 \end{aligned}$$

This is the volume of the approximating rectangular boxes shown in Figure 7. □

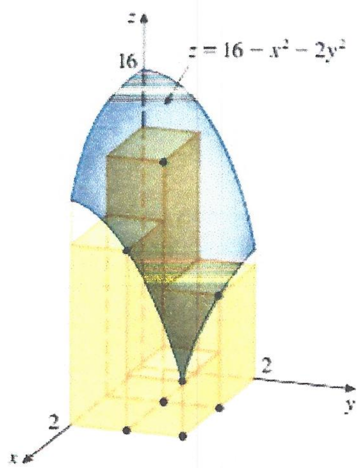


FIGURE 7

We get better approximations to the volume in Example 1 if we increase the number of squares. Figure 8 shows how the columns start to look more like the actual solid and the corresponding approximations become more accurate when we use 16, 64, and 256 squares. In the next section we will be able to show that the exact volume is 48.

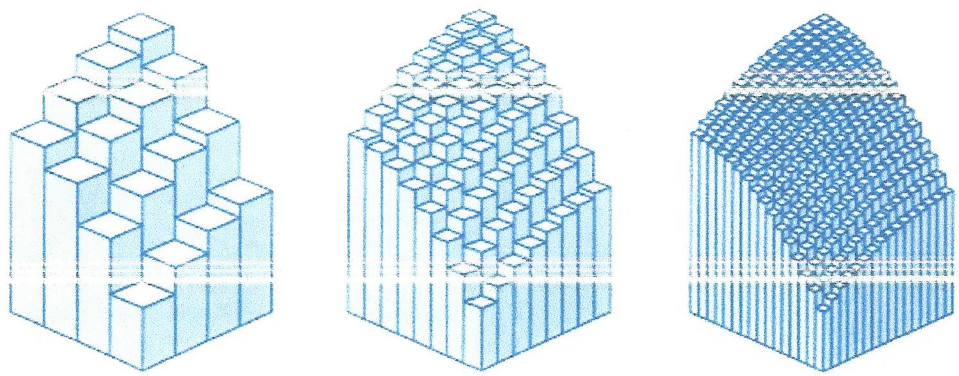


FIGURE 8
The Riemann sum approximations to the volume under $z = 16 - x^2 - 2y^2$ become more accurate as m and n increase.

(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

Riemann Sums

TOOLBOX:

$\sum_{k=1}^{n} h \cdot w$ $\quad h = f(x_k) \quad x_k = a + (\Delta x) \cdot k \quad w = \Delta x = \frac{b-a}{n}$

$$\sum_{k=1}^{n} f(x_k) \cdot \Delta x$$

APPROXIMATION

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k) \cdot \Delta x = \int_a^b f(x)\,dx$$

BETTER APPROXIMATION

EXAMPLE 1: Write $\int_1^5 (x^2 + 3)\,dx$ as a Riemann Sum.

EXAMPLE 2: Write $\lim_{n \to \infty} \sum_{k=1}^{n} \left(\sqrt{1 + \frac{3k}{n}} \cdot \frac{1}{n} \right)$ as a definite integral.

EXAMPLE 3: Which of the limits is equal to $\int_2^5 x^2\,dx$? *(from Big Book AB Test, p. 29 #30)*

(A) $\lim_{n \to \infty} \sum_{k=1}^{n} \left(2 + \frac{k}{n}\right)^2 \frac{1}{n}$

(B) $\lim_{n \to \infty} \sum_{k=1}^{n} \left(2 + \frac{k}{n}\right)^2 \frac{3}{n}$

(C) $\lim_{n \to \infty} \sum_{k=1}^{n} \left(2 + \frac{3k}{n}\right)^2 \frac{1}{n}$

(D) $\lim_{n \to \infty} \sum_{k=1}^{n} \left(2 + \frac{3k}{n}\right)^2 \frac{3}{n}$

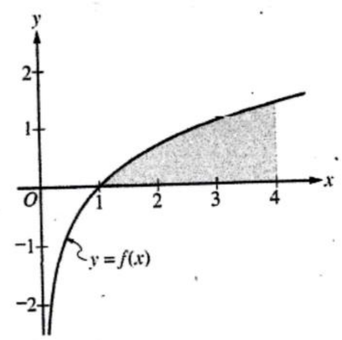
EXAMPLE 4: The function *f* is given by $f(x) = \ln x$. The graph of *f* is shown at right. Which of the following limits is equal to the area of the shaded region? *(from Big Book BC Test, p. 29 #29)*

(A) $\lim_{n \to \infty} \sum_{k=1}^{n} \left(1 + \ln\left(\frac{3k}{n}\right)\right) \frac{3}{n}$

(B) $\lim_{n \to \infty} \sum_{k=1}^{n} \ln\left(1 + \frac{3k}{n}\right) \frac{3}{n}$

(C) $\lim_{n \to \infty} \sum_{k=1}^{n} \ln\left(\frac{4}{n}\right)\left(1 + \frac{4k}{n}\right)$

(D) $\lim_{n \to \infty} \sum_{k=1}^{n} \ln\left(1 + \frac{4k}{n}\right) \frac{4}{n}$



16.1 DOUBLE INTEGRALS OVER RECTANGLES

In much the same way that our attempt to solve the area problem led to the definition of a definite integral, we now seek to find the volume of a solid and in the process we arrive at the definition of a double integral.

REVIEW OF THE DEFINITE INTEGRAL

First let's recall the basic facts concerning definite integrals of functions of a single variable. If $f(x)$ is defined for $a \leq x \leq b$, we start by dividing the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/n$ and we choose sample points x_i^* in these subintervals. Then we form the Riemann sum

$$\boxed{1} \qquad \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

and take the limit of such sums as $n \to \infty$ to obtain the definite integral of f from a to b:

$$\boxed{2} \qquad \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

In the special case where $f(x) \geq 0$, the Riemann sum can be interpreted as the sum of the areas of the approximating rectangles in Figure 1, and $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$ from a to b.

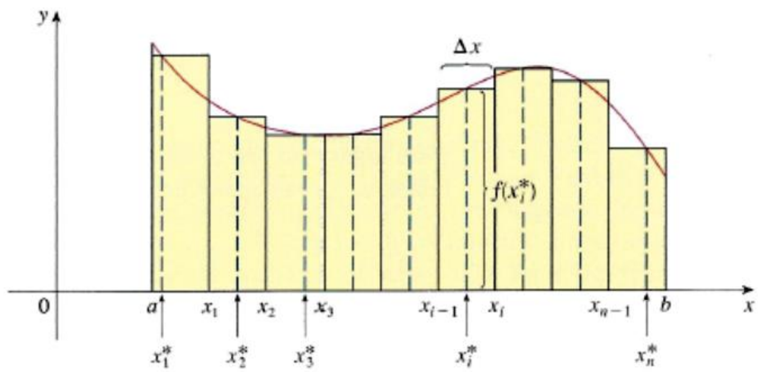


FIGURE 1

VOLUMES AND DOUBLE INTEGRALS

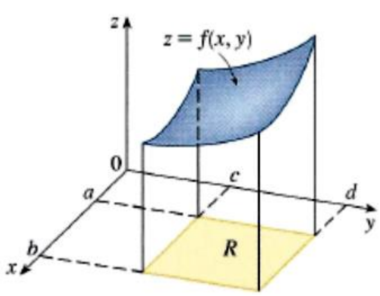


FIGURE 2

In a similar manner we consider a function f of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ c \leq y \leq d\}$$

and we first suppose that $f(x, y) \geq 0$. The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f, that is,

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq f(x, y),\ (x, y) \in R\}$$

(See Figure 2.) Our goal is to find the volume of S.

The first step is to divide the rectangle R into subrectangles. We accomplish this by dividing the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/m$ and dividing $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = (d - c)/n$. By draw-ing lines parallel to the coordinate axes through the endpoints of these subintervals, as in Figure 3, we form the subrectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leqslant x \leqslant x_i,\ y_{j-1} \leqslant y \leqslant y_j\}$$

each with area $\Delta A = \Delta x\, \Delta y$.

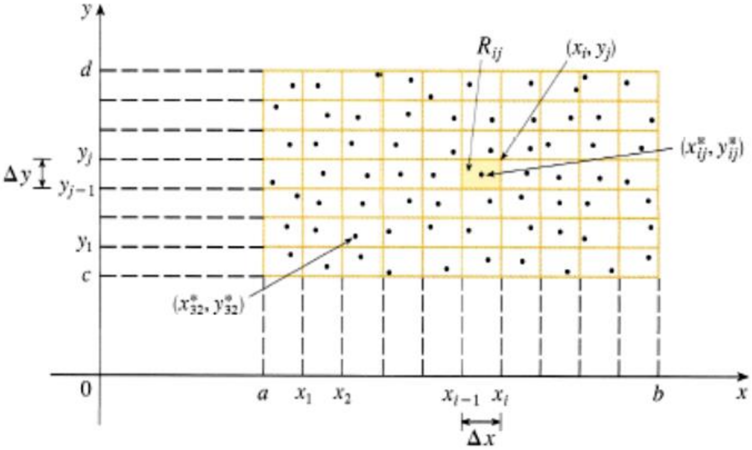


FIGURE 3
Dividing R into subrectangles

If we choose a **sample point** (x_{ij}^*, y_{ij}^*) in each R_{ij}, then we can approximate the part of S that lies above each R_{ij} by a thin rectangular box (or "column") with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$ as shown in Figure 4. (Compare with Figure 1.) The volume of this box is the height of the box times the area of the base rectangle:

$$f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

If we follow this procedure for all the rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volume of S:

3
$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

(See Figure 5.) This double sum means that for each subrectangle we evaluate f at the chosen point and multiply by the area of the subrectangle, and then we add the results.

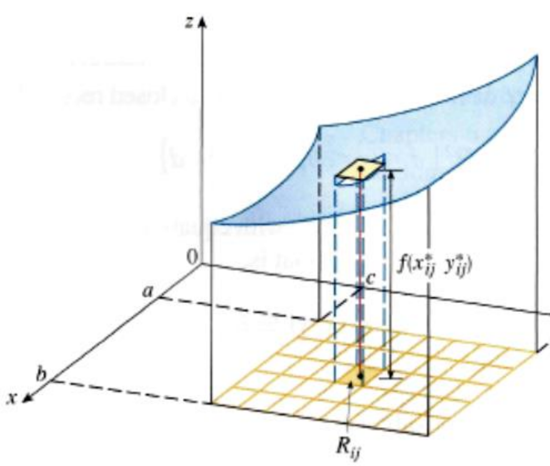


FIGURE 4

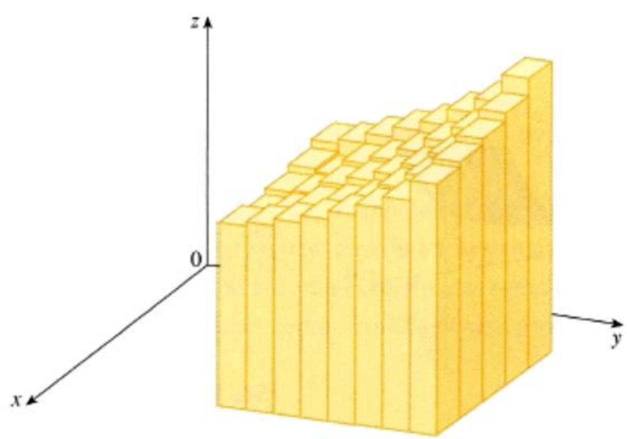


FIGURE 5

The sum in Definition 5,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \, \Delta A$$

is called a **double Riemann sum** and is used as an approximation to the value of the double integral. [Notice how similar it is to the Riemann sum in (1) for a function of a single variable.] If f happens to be a *positive* function, then the double Riemann sum represents the sum of volumes of columns, as in Figure 5, and is an approximation to the volume under the graph of f and above the rectangle R.

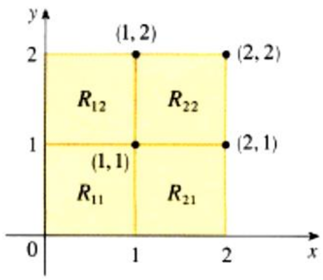


FIGURE 6

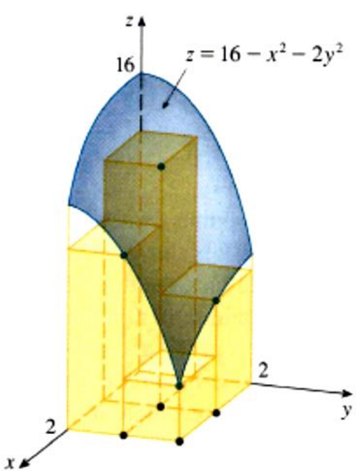


FIGURE 7

EXAMPLE 1 Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.

SOLUTION The squares are shown in Figure 6. The paraboloid is the graph of $f(x, y) = 16 - x^2 - 2y^2$ and the area of each square is 1. Approximating the volume by the Riemann sum with $m = n = 2$, we have

$$\begin{aligned} V &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j) \, \Delta A \\ &= f(1, 1) \, \Delta A + f(1, 2) \, \Delta A + f(2, 1) \, \Delta A + f(2, 2) \, \Delta A \\ &= 13(1) + 7(1) + 10(1) + 4(1) = 34 \end{aligned}$$

This is the volume of the approximating rectangular boxes shown in Figure 7. □

We get better approximations to the volume in Example 1 if we increase the number of squares. Figure 8 shows how the columns start to look more like the actual solid and the corresponding approximations become more accurate when we use 16, 64, and 256 squares. In the next section we will be able to show that the exact volume is 48.

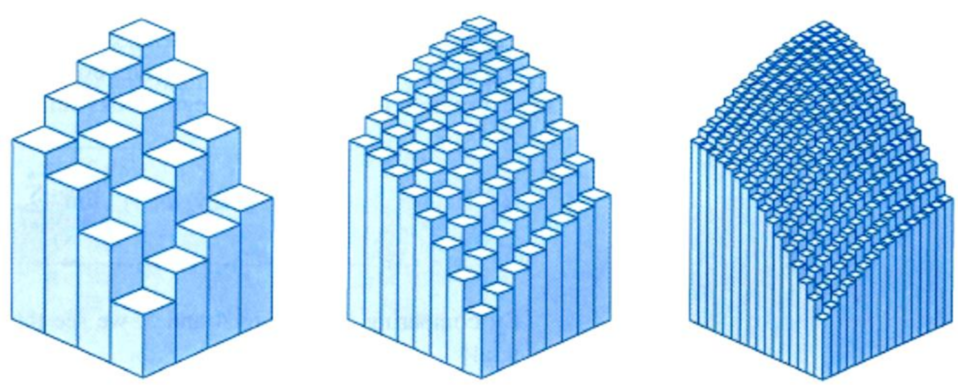


FIGURE 8
The Riemann sum approximations to the volume under $z = 16 - x^2 - 2y^2$ become more accurate as m and n increase.

(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

Volumes of Revolution: The Shell Method

- In the ***Disk and Washer Methods*** of finding volumes of solids, we revolved a region about the *x*- or *y*-axis. The representative rectangle was **perpendicular** to the axis of revolution. This rectangle was used to find the height of the slices before the revolution occurred, which in turn gave us the **radius** of the circular slice.

- If the revolution was about a **horizontal** axis of rotation, the slices cut off intervals on the *x*-axis, so the function was expressed in terms of *x*, the representative rectangle was perpendicular to the axis of rotation, and the limits of integration were along the *x*-axis, $[a,\ b]$.

- If the revolution was about a **vertical** axis of rotation, the slices cut off intervals on the *y*-axis, so the function was expressed in terms of *y*, *(the inverse of f(x))*, the representative rectangle was perpendicular to the axis of rotation, and the limits of integration were along the *y*-axis, $[c,\ d]$.

- When the revolution was about a vertical axis of rotation, the work got a little complicated because of the process of re-writing the function in terms of *y*. There is another method which is useful when performing this type of revolution, and it allows us to revolve about a vertical axis of rotation while leaving the function and limits of integration in terms of *x*. This method is called the ***Shell Method*****.** The **significant difference** between this method and the Disk and Washer Methods is that **the representative rectangle is *parallel* to the axis of rotation.** ***Therefore, if you want to revolve a region about a vertical axis of revolution, you will leave the function in terms of x, and you will integrate on the interval*** $[a,\ b]$**.** The Shell Method is also useful when the region you are revolving involves dividing it up because of its boundaries.

- A **cylindrical shell** is a solid enclosed by two concentric right circular cylinders. The volume of the shell can be found by finding the volume of the larger cylinder and subtracting the volume of the smaller cylinder. This method (requiring no calculus) is fine if there are no curved surfaces. In general, the volume of a shell with ***R*** representing the larger radius, and ***r*** representing the smaller radius, can be written as:

$$V = [\text{area of cross-section}] \cdot [\text{height}] \Rightarrow [\pi R^2 - \pi r^2] \cdot h$$

$$\Rightarrow \pi[R^2 - r^2]h \Rightarrow \pi[(R+r)(R-r)]h$$

Since the average radius of the shell is $\frac{1}{2}(R+r)$ and its thickness is $(R-r)$, this can be written as:

$$V = 2\pi\left[\frac{1}{2}(R+r)\right] \cdot h \cdot (R-r)$$

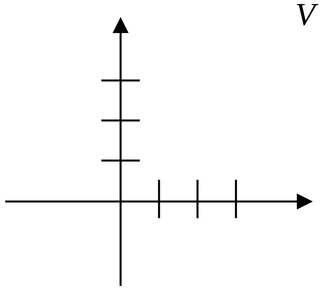
$$V = 2\pi[\text{average radius}] \cdot [\text{ height}] \cdot [\text{thickness}]$$

The Shell Method

$V = 2\pi\int_a^b p(x) \cdot f(x) \cdot dx$	$V = 2\pi\int_c^d p(y) \cdot f(y) \cdot dy$
Vertical Axis of Revolution	*Horizontal Axis of Revolution*

Note that $h(x)$ is often used for the height of the representative rectangle.

EXAMPLE 1: Consider the region bounded by the x-axis and $f(x) = (x-2)^2 + 1$ on $[2,\ 3]$. Find the volume of the solid obtained by revolving the region about the y-axis.



$$V = 2\pi\int_2^3 x\left[(x-2)^2 + 1\right] dx \quad \Rightarrow$$

$$\Rightarrow \quad 2\pi\int_2^3 x\left[(x^2 - 4x + 4) + 1\right] dx \quad \Rightarrow \quad 2\pi\int_2^3 (x^3 - 4x^2 + 5x)\, dx \quad \Rightarrow$$

$$\Rightarrow \quad 2\pi\left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{5x^2}{2}\right]_2^3 \quad \Rightarrow \quad 2\pi\left[\left(\frac{81}{4} - 36 + \frac{45}{2}\right) - \left(4 - \frac{32}{3} + 10\right)\right] \quad \Rightarrow$$

$$\Rightarrow \quad \frac{41\pi}{6} \text{ units}^3$$

- *If the region is bounded by two non-zero functions,* $f(x)$ and $g(x)$, with $f(x) \geq g(x)$, *the height is found by* $\left[f(x) - g(x)\right]$, *so the general formula becomes:*

$$V = 2\pi\int_a^b p(x)\cdot\left[f(x) - g(x)\right] dx$$

EXAMPLE 2: Let $f(x) = 3 - x^2$ and $g(x) = 3x - 1$ and let R be the region between the graphs of f and g on $[0,\ 1]$. Find the volume of the solid generated by revolving R about the y-axis.

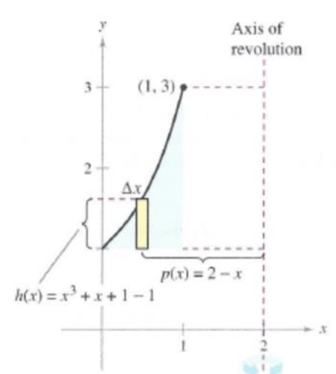
$$V = 2\pi\int_0^1 x\left[(3 - x^2) - (3x - 1)\right] dx \quad \text{since} \quad f\left(\frac{1}{2}\right) > g\left(\frac{1}{2}\right). \text{ Therefore:}$$

$$\Rightarrow \quad 2\pi\int_0^1 x\left[-x^2 - 3x + 4\right] dx \quad \Rightarrow \quad 2\pi\int_0^1 (-x^3 - 3x^2 + 4x)\, dx$$

$$\Rightarrow -2\pi\left[\frac{x^4}{4} + x^3 - 2x^2\right]_0^1 \quad \Rightarrow \quad -2\pi\left[\frac{1}{4} + 1 - 2\right] \quad \Rightarrow$$

$$\Rightarrow \quad -2\pi\left[-\frac{3}{4}\right] \quad \Rightarrow \quad \frac{3\pi}{2} \text{ units}^3$$

EXAMPLE 3: Find the volume of the solid generated by revolving the region bounded by the graphs of $y = x^3 + x + 1$, $y = 1$, and $x = 1$ about the line $x = 2$.



$$V = 2\pi\int_0^1 (2 - x)(x^3 + x + 1 - 1)\, dx$$

$$= 2\pi\int_0^1 (-x^4 + 2x^3 - x^2 + 2x)\, dx$$

$$= 2\pi\left[-\frac{x^5}{5} + \frac{x^4}{2} - \frac{x^3}{3} + x^2\right]_0^1$$

$$= 2\pi\left[-\frac{1}{5} + \frac{1}{2} - \frac{1}{3} + 1\right]$$

$$= \frac{29\pi}{15}$$

IV

Volumes of Revolution: The Shell Method

- In the ***Disk and Washer Methods*** of finding volumes of solids, we revolved a region about the x- or y-axis. The representative rectangle was **perpendicular** to the axis of revolution. This rectangle was used to find the height of the slices before the revolution occurred, which in turn gave us the **radius** of the circular slice.

- If the revolution was about a **horizontal** axis of rotation, the slices cut off intervals on the x-axis, so the function was expressed in terms of x, the representative rectangle was perpendicular to the axis of rotation, and the limits of integration were along the x-axis, $[a,\ b]$.

- If the revolution was about a **vertical** axis of rotation, the slices cut off intervals on the y-axis, so the function was expressed in terms of y, *(the inverse of f(x))*, the representative rectangle was perpendicular to the axis of rotation, and the limits of integration were along the y-axis, $[c,\ d]$.

- When the revolution was about a vertical axis of rotation, the work got a little complicated because of the process of re-writing the function in terms of y. There is another method which is useful when performing this type of revolution, and it allows us to revolve about a vertical axis of rotation while leaving the function and limits of integration in terms of x. This method is called the ***Shell Method***. The **significant difference** between this method and the Disk and Washer Methods is that **the representative rectangle is *parallel* to the axis of rotation.** ***Therefore, if you want to revolve a region about a vertical axis of revolution, you will leave the function in terms of x, and you will integrate on the interval*** $[a,\ b]$. The Shell Method is also useful when the region you are revolving involves dividing it up because of its boundaries.

- A **cylindrical shell** is a solid enclosed by two concentric right circular cylinders. The volume of the shell can be found by finding the volume of the larger cylinder and subtracting the volume of the smaller cylinder. This method (requiring no calculus) is fine if there are no curved surfaces. In general, the volume of a shell with R representing the larger radius, and r representing the smaller radius, can be written as:

$$V = [\text{area of cross-section}] \cdot [\text{height}] \Rightarrow [\pi R^2 - \pi r^2] \cdot h$$

$$\Rightarrow \pi[R^2 - r^2]h \Rightarrow \pi[(R+r)(R-r)]h$$

Since the average radius of the shell is $\frac{1}{2}(R+r)$ and its thickness is $(R-r)$, this can be written as:

$$V = 2\pi\left[\frac{1}{2}(R+r)\right] \cdot h \cdot (R-r)$$

$$V = 2\pi[\text{average radius}] \cdot [\text{ height}] \cdot [\text{thickness}]$$

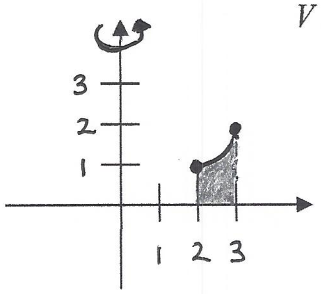
The Shell Method

$$V = 2\pi\int_a^b p(x) \cdot f(x) \cdot dx \qquad\qquad V = 2\pi\int_c^d p(y) \cdot f(y) \cdot dy$$

Vertical Axis of Revolution — *Horizontal Axis of Revolution*

Note that $h(x)$ is often used for the height of the representative rectangle.

EXAMPLE 1: Consider the region bounded by the x-axis and $f(x)=(x-2)^2+1$ on $[2,\ 3]$. Find the volume of the solid obtained by revolving the region about the y-axis.



$$V = 2\pi\int_2^3 x\left[(x-2)^2+1\right]dx \Rightarrow$$

$$\Rightarrow 2\pi\int_2^3 x\left[(x^2-4x+4)+1\right]dx \Rightarrow 2\pi\int_2^3 (x^3-4x^2+5x)\,dx \Rightarrow$$

$$\Rightarrow 2\pi\left[\frac{x^4}{4}-\frac{4x^3}{3}+\frac{5x^2}{2}\right]_2^3 \Rightarrow 2\pi\left[\left(\frac{81}{4}-36+\frac{45}{2}\right)-\left(4-\frac{32}{3}+10\right)\right] \Rightarrow$$

$$\Rightarrow \frac{41\pi}{6}\ \text{units}^3$$

- *If the region is bounded by two non-zero functions,* $f(x)$ and $g(x)$, with $f(x)\ge g(x)$, *the height is found by* $[f(x)-g(x)]$, *so the general formula becomes:*

$$V=2\pi\int_a^b p(x)\cdot[f(x)-g(x)]\,dx$$

EXAMPLE 2: Let $f(x)=3-x^2$ and $g(x)=3x-1$ and let R be the region between the graphs of f and g on $[0,\ 1]$. Find the volume of the solid generated by revolving R about the y-axis.

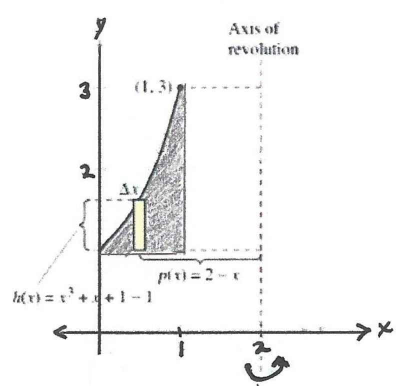
$$V=2\pi\int_0^1 x\left[(3-x^2)-(3x-1)\right]dx \quad \text{since } f\left(\frac{1}{2}\right) > g\left(\frac{1}{2}\right). \text{ Therefore:}$$

$$\Rightarrow 2\pi\int_0^1 x\left[-x^2-3x+4\right]dx \Rightarrow 2\pi\int_0^1 (-x^3-3x^2+4x)\,dx$$

$$\Rightarrow -2\pi\left[\frac{x^4}{4}+x^3-2x^2\right]_0^1 \Rightarrow -2\pi\left[\frac{1}{4}+1-2\right] \Rightarrow$$

$$\Rightarrow -2\pi\left[-\frac{3}{4}\right] \Rightarrow \frac{3\pi}{2}\ \text{units}^3$$

EXAMPLE 3: Find the volume of the solid generated by revolving the region bounded by the graphs of $y=x^3+x+1$, $y=1$, and $x=1$ about the line $x=2$.



$$V = 2\pi\int_0^1 (2-x)(x^3+x+1-1)\,dx$$

$$= 2\pi\int_0^1 (-x^4+2x^3-x^2+2x)\,dx$$

$$= 2\pi\left[-\frac{x^5}{5}+\frac{x^4}{2}-\frac{x^3}{3}+x^2\right]_0^1$$

$$= 2\pi\left[-\frac{1}{5}+\frac{1}{2}-\frac{1}{3}+1\right]$$

$$= \frac{29\pi}{15}$$

I

Conic Sections

- A parabola is the set of points in a plane equidistant from a fixed point *(called the **focus**)* and a fixed line *(called the **directrix**)*.

Standard Form of a Parabola

HORIZONTAL	VERTICAL
$(y-k)^2 = 4p(x-h)$	$(x-h)^2 = 4p(y-k)$

General Form of the equation of a Parabola

HORIZONTAL:	VERTICAL:
$y^2 + Cx + Dy + E = 0$	$x^2 + Cx + Dy + E = 0$

A parabola opens in the direction of the non-squared term!

- An ellipse is the set of all points in a plane whose distances from two fixed points *(called the foci)* is a constant sum.

Standard Form of an Ellipse

*The equations **must** be equal to 1.*

$a > b$

HORIZONTAL	VERTICAL
$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$
Major Axis is parallel to the x-axis.	*Major Axis is parallel to the y-axis.*

General Form of an equation of an Ellipse:

$$Ax^2 + By^2 + Cx + Dy + E = 0$$

Note: the x^2 and y^2 terms are both positive, but A and B are different!

- A hyperbola is the set of all points in a plane such that the absolute value of the differences of the distances from two fixed points *(called the foci)* is a constant.

Standard Form of a Hyperbola

*The equations **must** be equal to 1.*

***"a"** is in the positive term*

HORIZONTAL	VERTICAL
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Transverse Axis is parallel to the x-axis.	*Transverse Axis is parallel to the y-axis.*

General Form of an equation of a Hyperbola:

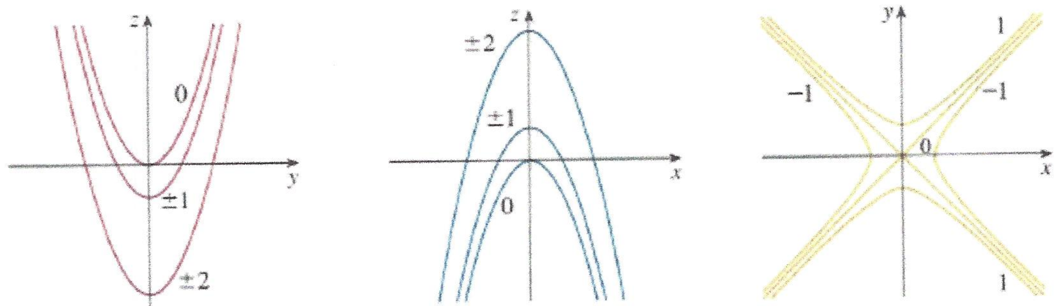
$$Ax^2 + By^2 + Cx + Dy + E = 0$$

Note: A and B have different signs!

Why do we care?

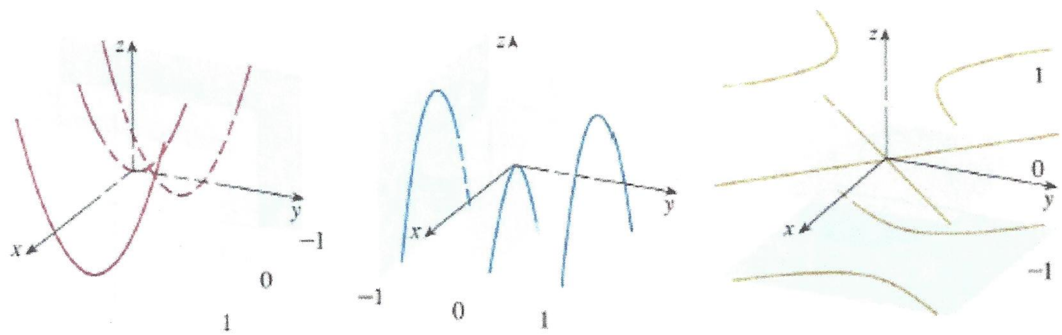
EXAMPLE 5 Sketch the surface $z = y^2 - x^2$.

SOLUTION The traces in the vertical planes $x = k$ are the parabolas $z = y^2 - k^2$, which open upward. The traces in $y = k$ are the parabolas $z = -x^2 + k^2$, which open downward. The horizontal traces are $y^2 - x^2 = k$, a family of hyperbolas. We draw the families of traces in Figure 6, and we show how the traces appear when placed in their correct planes in Figure 7.



Traces in $x = k$ are $z = y^2 - k^2$ Traces in $y = k$ are $z = -x^2 + k^2$ Traces in $z = k$ are $y^2 - x^2 = k$

FIGURE 6
Vertical traces are parabolas; horizontal traces are hyperbolas. All traces are labeled with the value of k.



Traces in $x = k$ Traces in $y = k$ Traces in $z = k$

FIGURE 7
Traces moved to their correct planes

TEC In Module 13.6A you can investigate how traces determine the shape of a surface.

In Figure 8 we fit together the traces from Figure 7 to form the surface $z = y^2 - x^2$, a **hyperbolic paraboloid**. Notice that the shape of the surface near the origin resembles that of a saddle. This surface will be investigated further in Section 15.7 when we discuss saddle points.

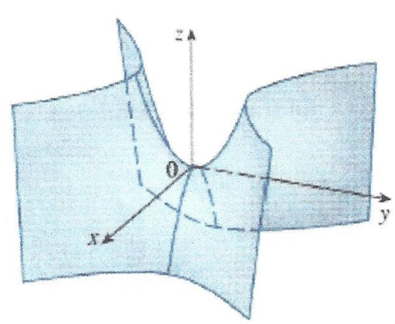
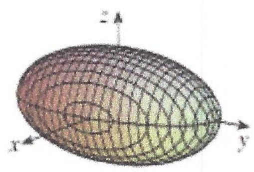
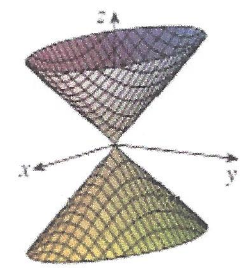
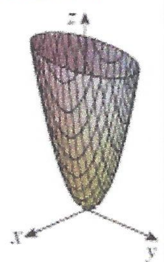
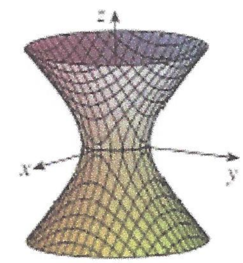
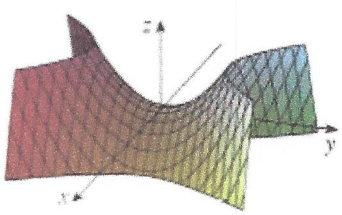
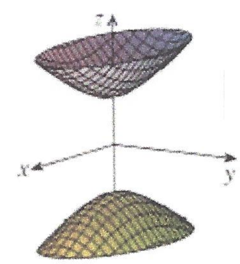


FIGURE 8
The surface $z = y^2 - x^2$ is a hyperbolic paraboloid.

□

(V)

TABLE 1 Graphs of quadric surfaces

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere. (no minus sign)	Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k=0$. (no linear terms)
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid. (one linear term; 2 quadratic terms w/ same sign)	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative. (one minus sign)
Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated. (one linear term; 2 quadratic terms w/ opposite signs)	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets. (two minus signs)

Ⓥ p.4

TEC In Module 13.6B you can see how changing a, b, and c in Table 1 affects the shape of the quadric surface.

EXAMPLE 7 Identify and sketch the surface $4x^2 - y^2 + 2z^2 + 4 = 0$.

SOLUTION Dividing by -4, we first put the equation in standard form:

$$-x^2 + \frac{y^2}{4} - \frac{z^2}{2} = 1$$

Comparing this equation with Table 1, we see that it represents a hyperboloid of two sheets, the only difference being that in this case the axis of the hyperboloid is the y-axis. The traces in the xy- and yz-planes are the hyperbolas

$$-x^2 + \frac{y^2}{4} = 1 \qquad z = 0 \qquad \text{and} \qquad \frac{y^2}{4} - \frac{z^2}{2} = 1 \qquad x = 0$$

The surface has no trace in the xz-plane, but traces in the vertical planes $y = k$ for $|k| > 2$ are the ellipses

$$x^2 + \frac{z^2}{2} = \frac{k^2}{4} - 1 \qquad y = k$$

which can be written as

$$\frac{x^2}{\dfrac{k^2}{4} - 1} + \frac{z^2}{2\left(\dfrac{k^2}{4} - 1\right)} = 1 \qquad y = k$$

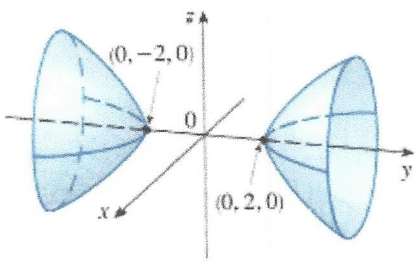


FIGURE 10
$4x^2 - y^2 + 2z^2 + 4 = 0$

These traces are used to make the sketch in Figure 10. □

EXAMPLE 8 Classify the quadric surface $x^2 + 2z^2 - 6x - y + 10 = 0$.

SOLUTION By completing the square we rewrite the equation as

$$y - 1 = (x - 3)^2 + 2z^2$$

Comparing this equation with Table 1, we see that it represents an elliptic paraboloid. Here, however, the axis of the paraboloid is parallel to the y-axis, and it has been shifted so that its vertex is the point $(3, 1, 0)$. The traces in the plane $y = k$ $(k > 1)$ are the ellipses

$$(x - 3)^2 + 2z^2 = k - 1 \qquad y = k$$

The trace in the xy-plane is the parabola with equation $y = 1 + (x - 3)^2$, $z = 0$. The paraboloid is sketched in Figure 11.

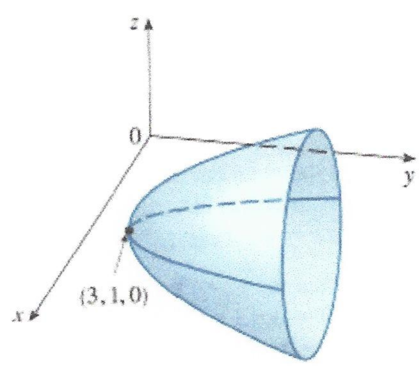


FIGURE 11
$x^2 + 2z^2 - 6x - y + 10 = 0$

□

(V) P.5

21–28 Match the equation with its graph (labeled I–VIII). Give reasons for your choices.

21. $x^2 + 4y^2 + 9z^2 = 1$

22. $9x^2 + 4y^2 + z^2 = 1$

23. $x^2 - y^2 + z^2 = 1$

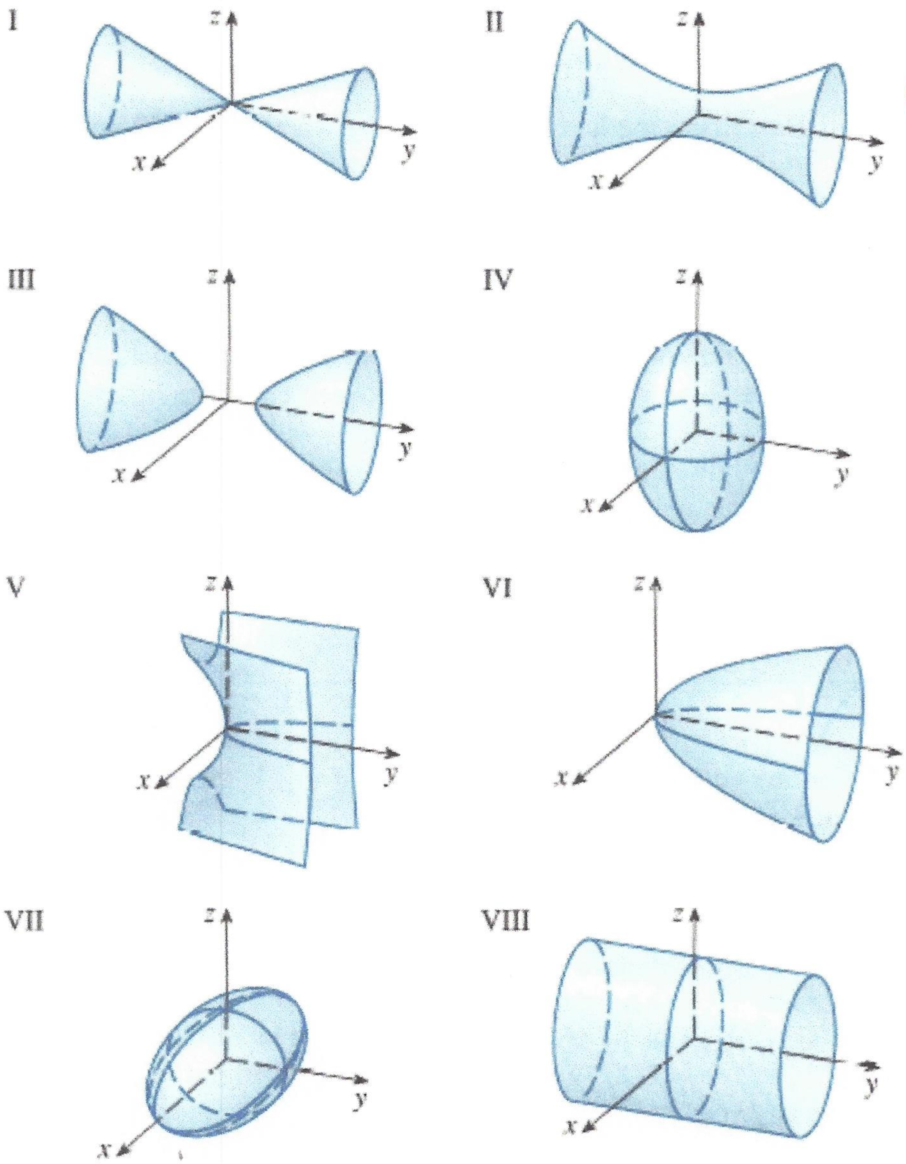
24. $-x^2 + y^2 - z^2 = 1$

25. $y = 2x^2 + z^2$

26. $y^2 = x^2 + 2z^2$

27. $x^2 + 2z^2 = 1$

28. $y = x^2 - z^2$



(21) VII
(22) IV
(23) II
(24) III
(25) VI
(26) I
(27) VIII

(21) $x^2 + \frac{y^2}{(\frac{1}{2})^2} + \frac{z^2}{(\frac{1}{3})^2} = 1$

x-int @ ± 1

y-int @ $\pm \frac{1}{2}$

z-int @ $\pm \frac{1}{3}$

(22) $\frac{x^2}{(\frac{1}{3})^2} + \frac{y^2}{(\frac{1}{2})^2} + z^2 = 1$

x-int: $\pm \frac{1}{3}$

y-int: $\pm \frac{1}{2}$

z-int: ± 1

(23) $a=b=c=1$
$-y^2$ so axis of hyper. is y-axis (one sheet)

(24) $a=b=c=1$
does not inter. xz-plane
axis is y-axis

(25) • no x, z that satisfy eq. for $y<0$.
• intersects $y=k>0$
• non-squared $y \Rightarrow$ paraboloid

(26) • eq. of cone
• axis is y-axis

(27) • y missing
• cylinder

(28) • hyperbolic paraboloid
• ellipse in xz-plane

Ⅰ

p.6

23. A fuel-efficient way to travel from earth to Mars is to follow a semi-elliptical orbit known as a *Hohmann transfer orbit.*[2] The spacecraft leaves the earth at the point on the ellipse closest to the sun, and arrives at Mars at the point on the ellipse farthest from the sun, as shown in Figure 12.25. Let r_e be the radius of earth's orbit, and r_m the radius of Mars' orbit, and let $2a$ and $2b$ be the respective lengths of the horizontal and vertical axes of the ellipse.

(a) Orienting the ellipse as shown in the figure, with the sun at the origin, find a formula for this orbit in terms of r_e, r_m, and b.
(b) It can be shown that $b^2 = 2ar_e - r_e^2$. Given this and your answer to part (a), find a formula for b in terms of r_m and r_e.

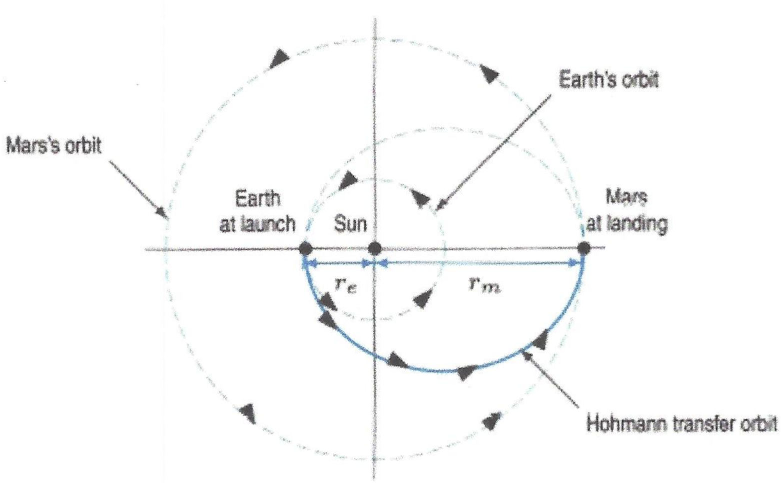


Figure 12.25

a) $\left(\dfrac{2x - r_m + r_e}{r_e + r_m}\right)^2 + \dfrac{y^2}{b^2} = 1$

b) $b = \sqrt{r_e r_m}$

[2]*Scientific American*, www.sciam.com, March 17, 2000. Note that in an actual Hohmann transfer, the spacecraft would begin in low earth orbit, not from the earth's surface. Timing is critical in order for the two planets to be correctly aligned. Calculations show that at launch, Mars must lead earth by about 45 degrees, which happens only once every 26 months. Note also that the orbits of earth and Mars are actually themselves elliptical, though here we treat them as circular.

(V)

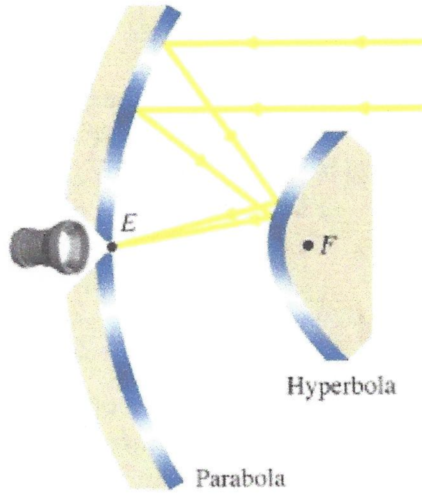


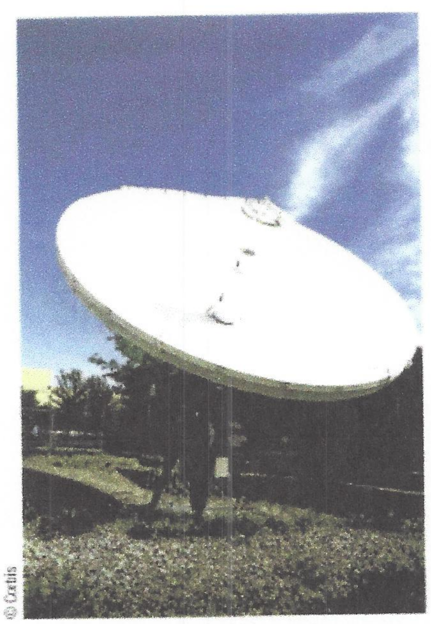
FIGURE 9.67
A combination of parabolic and hyperbolic mirrors

APPLICATIONS OF QUADRIC SURFACES

Examples of quadric surfaces can be found in the world around us. In fact, the world itself is a good example. Although the earth is commonly modeled as a sphere, a more accurate model is an ellipsoid because the earth's rotation has caused a flattening at the poles. (See Exercise 47.)

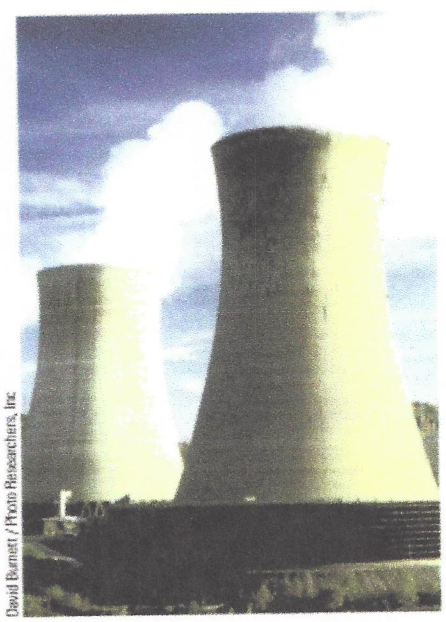
Circular paraboloids, obtained by rotating a parabola about its axis, are used to collect and reflect light, sound, and radio and television signals. In a radio telescope, for instance, signals from distant stars that strike the bowl are reflected to the receiver at the focus and are therefore amplified. (The idea is explained in Problem 16 on page 202.) The same principle applies to microphones and satellite dishes in the shape of paraboloids.

Cooling towers for nuclear reactors are usually designed in the shape of hyperboloids of one sheet for reasons of structural stability. Pairs of hyperboloids are used to transmit rotational motion between skew axes. (The cogs of gears are the generating lines of the hyperboloids. See Exercise 49.)



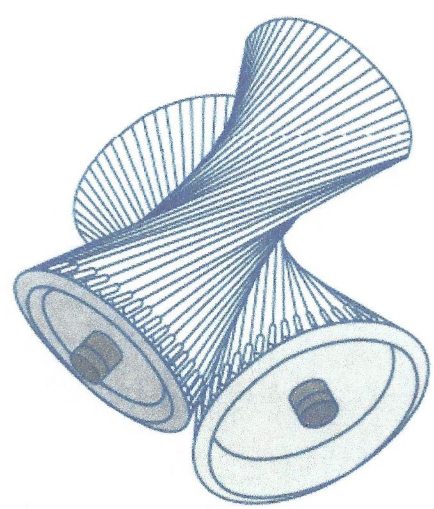
© Corbis

A satellite dish reflects signals to the focus of a paraboloid.



David Burnett / Photo Researchers, Inc

Nuclear reactors have cooling towers in the shape of hyperboloids.



Hyperboloids produce gear transmission.

Conic Sections

- A parabola is the set of points in a plane equidistant from a fixed point *(called the **focus**)* and a fixed line *(called the **directrix**)*.

Standard Form of a Parabola

HORIZONTAL	VERTICAL
$(y-k)^2 = 4p(x-h)$	$(x-h)^2 = 4p(y-k)$

General Form of the equation of a Parabola

HORIZONTAL:	VERTICAL:
$y^2 + Cx + Dy + E = 0$	$x^2 + Cx + Dy + E = 0$

A parabola opens in the direction of the non-squared term!

- An ellipse is the set of all points in a plane whose distances from two fixed points *(called the foci)* is a constant sum.

Standard Form of an Ellipse

*The equations **must** be equal to 1.*

$a > b$

HORIZONTAL	VERTICAL
$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$
Major Axis is parallel to the x-axis.	*Major Axis is parallel to the y-axis.*

General Form of an equation of an Ellipse:

$$Ax^2 + By^2 + Cx + Dy + E = 0$$

Note: the x^2 and y^2 terms are both positive, but A and B are different!

- A hyperbola is the set of all points in a plane such that the absolute value of the differences of the distances from two fixed points *(called the foci)* is a constant.

Standard Form of a Hyperbola

*The equations **must** be equal to 1.*

"a" is *in the positive term*

HORIZONTAL	VERTICAL
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Transverse Axis is parallel to the x-axis.	*Transverse Axis is parallel to the y-axis.*

General Form of an equation of a Hyperbola:

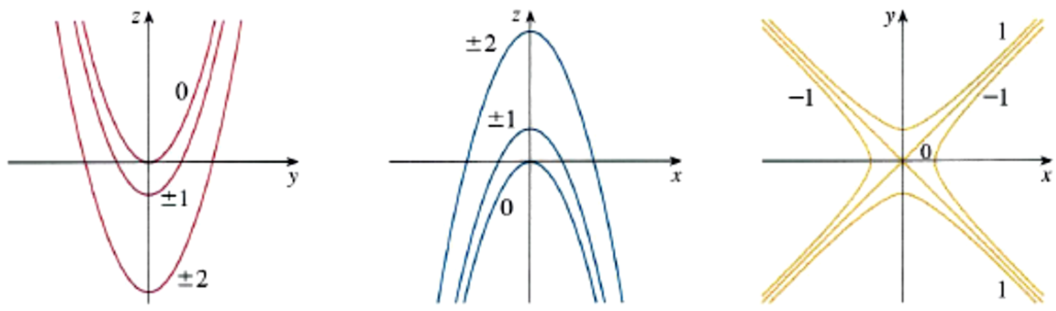
$$Ax^2 + By^2 + Cx + Dy + E = 0$$

Note: A and B have different signs!

Why do we care?

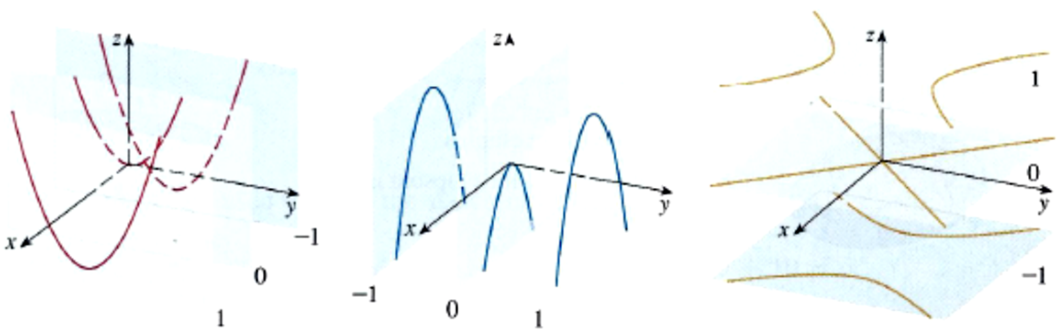
EXAMPLE 5 Sketch the surface $z = y^2 - x^2$.

SOLUTION The traces in the vertical planes $x = k$ are the parabolas $z = y^2 - k^2$, which open upward. The traces in $y = k$ are the parabolas $z = -x^2 + k^2$, which open downward. The horizontal traces are $y^2 - x^2 = k$, a family of **hyperbolas**. We draw the families of traces in Figure 6, and we show how the traces appear when placed in their correct planes in Figure 7.



Traces in $x = k$ are $z = y^2 - k^2$ | Traces in $y = k$ are $z = -x^2 + k^2$ | Traces in $z = k$ are $y^2 - x^2 = k$

FIGURE 6
Vertical traces are parabolas; horizontal traces are hyperbolas. All traces are labeled with the value of k.



Traces in $x = k$ | Traces in $y = k$ | Traces in $z = k$

FIGURE 7
Traces moved to their correct planes

TEC In Module 13.6A you can investigate how traces determine the shape of a surface.

In Figure 8 we fit together the traces from Figure 7 to form the surface $z = y^2 - x^2$, a **hyperbolic paraboloid**. Notice that the shape of the surface near the origin resembles that of a saddle. This surface will be investigated further in Section 15.7 when we discuss saddle points.

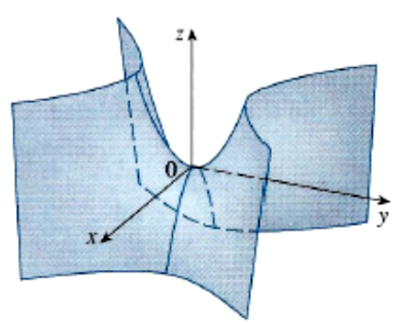
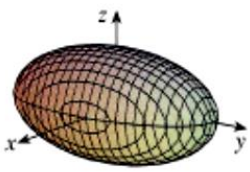
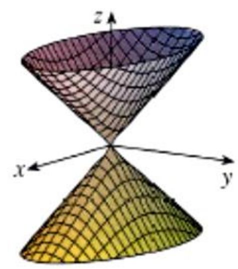
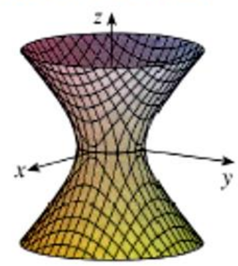
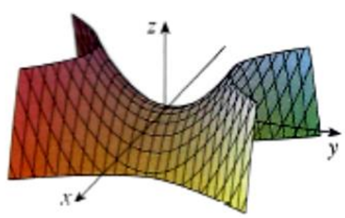
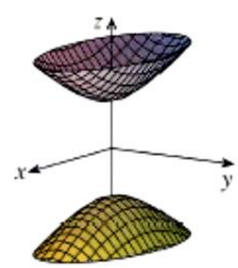


FIGURE 8
The surface $z = y^2 - x^2$ is a hyperbolic paraboloid.

□

TABLE 1 Graphs of quadric surfaces

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

TEC In Module 13.6B you can see how changing a, b, and c in Table 1 affects the shape of the quadric surface.

V EXAMPLE 7 Identify and sketch the surface $4x^2 - y^2 + 2z^2 + 4 = 0$.

SOLUTION Dividing by -4, we first put the equation in standard form:

$$-x^2 + \frac{y^2}{4} - \frac{z^2}{2} = 1$$

Comparing this equation with Table 1, we see that it represents a hyperboloid of two sheets, the only difference being that in this case the axis of the hyperboloid is the y-axis. The traces in the xy- and yz-planes are the hyperbolas

$$-x^2 + \frac{y^2}{4} = 1 \qquad z = 0 \qquad \text{and} \qquad \frac{y^2}{4} - \frac{z^2}{2} = 1 \qquad x = 0$$

The surface has no trace in the xz-plane, but traces in the vertical planes $y = k$ for $|k| > 2$ are the ellipses

$$x^2 + \frac{z^2}{2} = \frac{k^2}{4} - 1 \qquad y = k$$

which can be written as

$$\frac{x^2}{\frac{k^2}{4} - 1} + \frac{z^2}{2\left(\frac{k^2}{4} - 1\right)} = 1 \qquad y = k$$

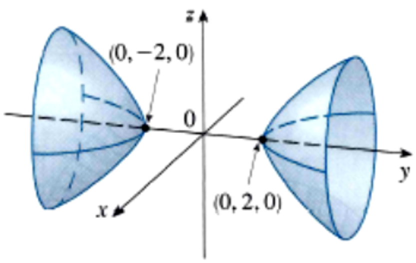


FIGURE 10
$4x^2 - y^2 + 2z^2 + 4 = 0$

These traces are used to make the sketch in Figure 10. □

EXAMPLE 8 Classify the quadric surface $x^2 + 2z^2 - 6x - y + 10 = 0$.

SOLUTION By completing the square we rewrite the equation as

$$y - 1 = (x - 3)^2 + 2z^2$$

Comparing this equation with Table 1, we see that it represents an elliptic paraboloid. Here, however, the axis of the paraboloid is parallel to the y-axis, and it has been shifted so that its vertex is the point $(3, 1, 0)$. The traces in the plane $y = k$ $(k > 1)$ are the ellipses

$$(x - 3)^2 + 2z^2 = k - 1 \qquad y = k$$

The trace in the xy-plane is the parabola with equation $y = 1 + (x - 3)^2$, $z = 0$. The paraboloid is sketched in Figure 11.

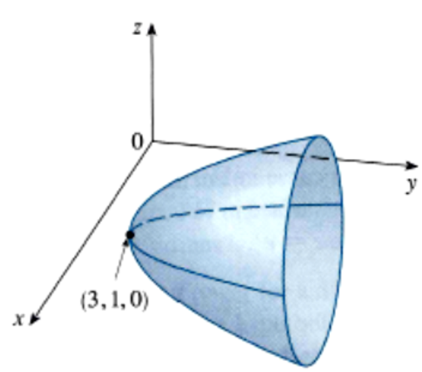


FIGURE 11
$x^2 + 2z^2 - 6x - y + 10 = 0$

□

21–28 Match the equation with its graph (labeled I–VIII). Give reasons for your choices.

21. $x^2 + 4y^2 + 9z^2 = 1$

22. $9x^2 + 4y^2 + z^2 = 1$

23. $x^2 - y^2 + z^2 = 1$

24. $-x^2 + y^2 - z^2 = 1$

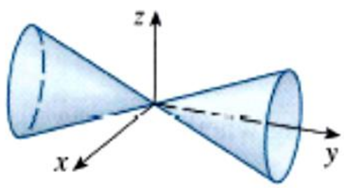
25. $y = 2x^2 + z^2$

26. $y^2 = x^2 + 2z^2$

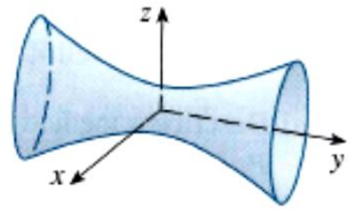
27. $x^2 + 2z^2 = 1$

28. $y = x^2 - z^2$

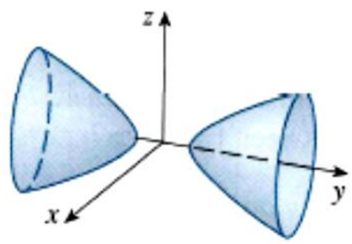
I



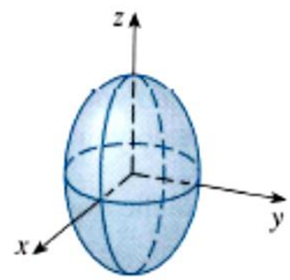
II



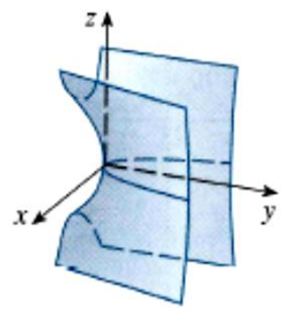
III



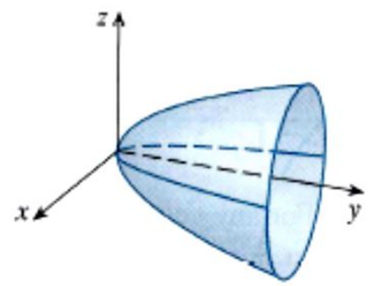
IV



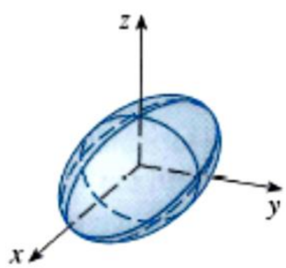
V



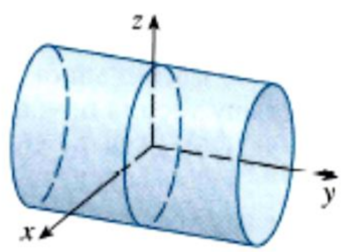
VI



VII



VIII



23. A fuel-efficient way to travel from earth to Mars is to follow a semi-elliptical orbit known as a *Hohmann transfer orbit.*[2] The spacecraft leaves the earth at the point on the ellipse closest to the sun, and arrives at Mars at the point on the ellipse farthest from the sun, as shown in Figure 12.25. Let r_e be the radius of earth's orbit, and r_m the radius of Mars' orbit, and let $2a$ and $2b$ be the respective lengths of the horizontal and vertical axes of the ellipse.

(a) Orienting the ellipse as shown in the figure, with the sun at the origin, find a formula for this orbit in terms of r_e, r_m, and b.

(b) It can be shown that $b^2 = 2ar_e - r_e^2$. Given this and your answer to part (a), find a formula for b in terms of r_m and r_e.

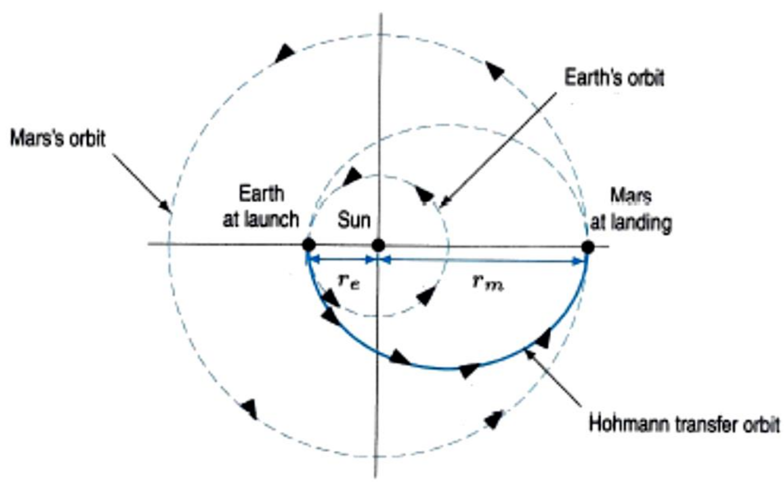


Figure 12.25

a) $$\left(\frac{2x - r_m + r_e}{r_e + r_m}\right)^2 + \frac{y^2}{b^2} = 1$$

b) $$b = \sqrt{r_e\, r_m}$$

[2] *Scientific American*, www.sciam.com, March 17, 2000. Note that in an actual Hohmann transfer, the spacecraft would begin in low earth orbit, not from the earth's surface. Timing is critical in order for the two planets to be correctly aligned. Calculations show that at launch, Mars must lead earth by about 45 degrees, which happens only once every 26 months. Note also that the orbits of earth and Mars are actually themselves elliptical, though here we treat them as circular.

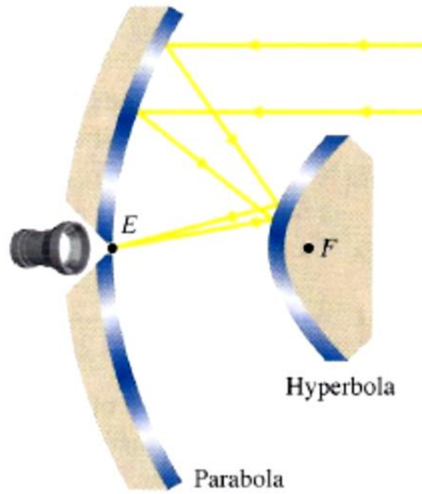


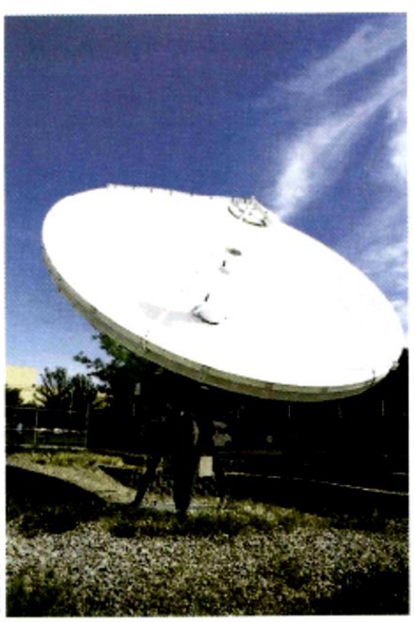
FIGURE 9.67
A combination of parabolic and hyperbolic mirrors

APPLICATIONS OF QUADRIC SURFACES

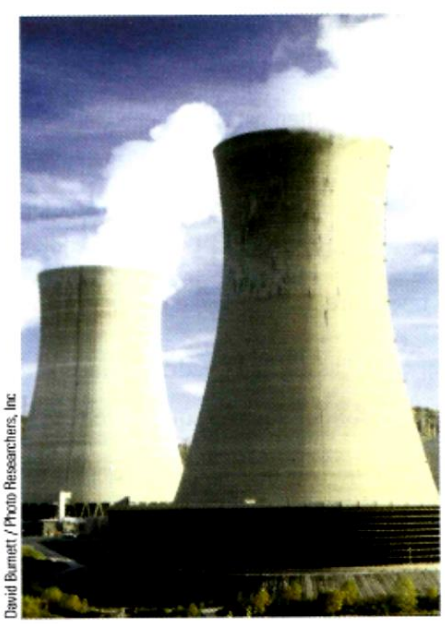
Examples of quadric surfaces can be found in the world around us. In fact, the world itself is a good example. Although the earth is commonly modeled as a sphere, a more accurate model is an ellipsoid because the earth's rotation has caused a flattening at the poles. (See Exercise 47.)

Circular paraboloids, obtained by rotating a parabola about its axis, are used to collect and reflect light, sound, and radio and television signals. In a radio telescope, for instance, signals from distant stars that strike the bowl are reflected to the receiver at the focus and are therefore amplified. (The idea is explained in Problem 16 on page 202.) The same principle applies to microphones and satellite dishes in the shape of paraboloids.

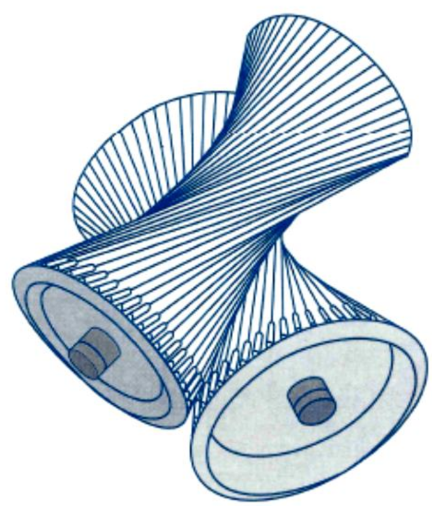
Cooling towers for nuclear reactors are usually designed in the shape of hyperboloids of one sheet for reasons of structural stability. Pairs of hyperboloids are used to transmit rotational motion between skew axes. (The cogs of gears are the generating lines of the hyperboloids. See Exercise 49.)



A satellite dish reflects signals to the focus of a paraboloid.



Nuclear reactors have cooling towers in the shape of hyperboloids.

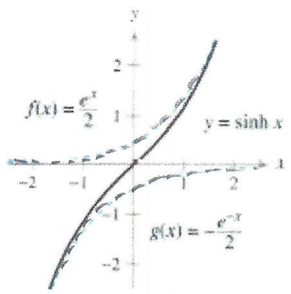


Hyperboloids produce gear transmission.

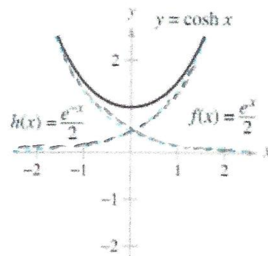
VI

Hyperbolic Functions

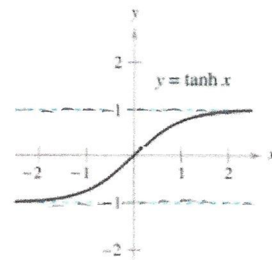
There is a special class of the even and odd combinations of the exponential functions e^x and e^{-x} which occur so frequently they are given a special name: **hyperbolic functions**. They have the same relationship to the hyperbola that trigonometric functions have to the circle.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



Domain: $(-\infty, \infty)$
Range: $[1, \infty)$



Domain: $(-\infty, \infty)$
Range: $(-1, 1)$

$$\sinh x = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x} = \frac{e^{x} - e^{-x}}{2} \qquad \cosh x = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x} = \frac{e^{x} + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

Note that the graphs of $\sinh x$ and $\cosh x$ can be obtained by the addition of ordinates using the exponential functions $y = \frac{1}{2}e^{x}$ and $y = -\frac{1}{2}e^{-x}$, while $\tanh x$ is obtained from the ratio of $\frac{\sinh x}{\cosh x}$.

Definitions of Hyperbolic Functions

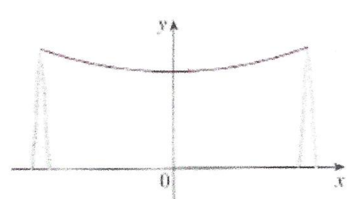
$$\sinh x = \frac{e^{x} - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

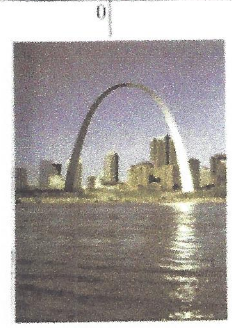
$$\cosh x = \frac{e^{x} + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

- Some mathematical applications of hyperbolic functions occur in science and engineering when an entity like light, velocity, electricity, or radioactivity is gradually absorbed or extinguished.

- The most common application is the use of hyperbolic cosine to describe the shape of a hanging wire. It can be shown that if a heavy flexible cable is suspended between two points at the same height, it takes the shape of a curve with equation $y = c + a\cosh\left(\frac{x}{a}\right)$, called a *catenary* (from the Latin word *catena*, which means "chain").

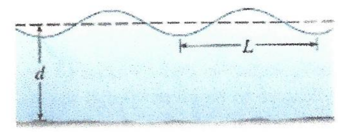




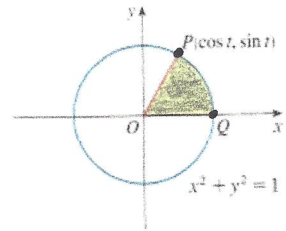
- The Gateway Arch in St. Louis is a structure designed using a hyperbolic cosine function.

VI. p.2

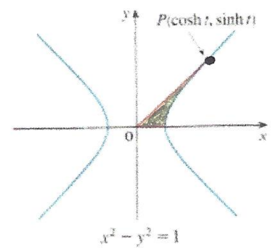
- Another hyperbolic function application occurs in the description of ocean waves. The velocity of a water wave with length L moving across a body of water with depth d is modeled by the function $v = \sqrt{\frac{gL}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)}$ where g is the acceleration due to gravity.



Recall that if t is any real number, then the point $P(\cos t, \sin t)$ lies on the unit circle $x^2 + y^2 = 1$ because $\cos^2 t + \sin^2 t = 1$, and t can be interpreted as the radian measure of $\angle POQ$, and it represents twice the area of the shaded circular region. This is why trigonometric functions are also called circular functions.



Likewise, if t is any real number, then the point $P(\cosh t, \sinh t)$ lies on the right branch of the hyperbola because $\cosh^2 t - \sinh^2 t = 1$ and $\cosh t \geq 1$. This time, t does not represent the measure of an angle, but it does represent twice the area of the shaded hyperbolic sector.



- Hyperbolic functions provide the ability of using hyperbolic substitutions instead of trigonometric substitutions for radical expressions, and they sometimes lead to simpler answers. However, trigonometric substitutions are more commonly used because trigonometric identities are more familiar than hyperbolic identities.

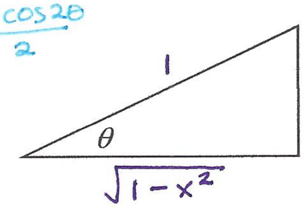
Consider $\int_{-1}^{1}\sqrt{1-x^2}\,dx$. This requires integration by parts $\left(\int u\,dv = uv - \int v\,du\right)$ and trigonometric substitution.

$$\int\sqrt{1-x^2}\,dx = x\sqrt{1-x^2} - \int\frac{-x^2}{\sqrt{1-x^2}}\,dx \qquad u=\sqrt{1-x^2},\ du = \frac{-x}{\sqrt{1-x^2}}dx,\ dv = dx,\ \text{and } v = x$$

$$= x\sqrt{1-x^2} + \int\frac{x^2}{\sqrt{1-x^2}}\,dx \qquad x=\sin\theta,\ dx=\cos\theta\,d\theta,\ \text{and } \sqrt{1-x^2}=\sqrt{1-\sin^2\theta}=\sqrt{\cos^2\theta}=\cos\theta$$

$$= x\sqrt{1-x^2} + \int\frac{\sin^2\theta}{\cos\theta}(\cos\theta\,d\theta)$$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$



$$= x\sqrt{1-x^2} + \int\frac{1-\cos 2\theta}{2}\,d\theta$$

$$= x\sqrt{1-x^2} + \frac{1}{2}\int 1\,d\theta - \frac{1}{2}\cdot\frac{1}{2}\int\cos 2\theta(2\,d\theta)$$

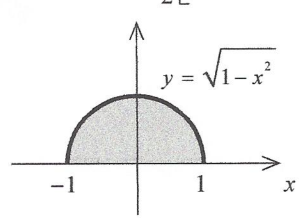
$\sin\theta = x$

$$= x\sqrt{1-x^2} + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C \qquad \theta=\sin^{-1}x,\ \sin 2\theta = 2\sin\theta\cos\theta$$

$$= x\sqrt{1-x^2} - \frac{1}{2}\left[\sin^{-1}x - \sin\theta\cos\theta\right] + C \Rightarrow x\sqrt{1-x^2} - \frac{1}{2}\left[\sin^{-1}x - x\sqrt{1-x^2}\right] + C$$

$$\int\sqrt{1-x^2}\,dx = \frac{1}{2}\left[x\sqrt{1-x^2} + \sin^{-1}x\right] + C$$

Hence, $\int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{1}{2}\left[x\sqrt{1-x^2} + \sin^{-1}x\right]_{-1}^{1} = \frac{\pi}{2}$



VI

Many of the trigonometric identities have corresponding hyperbolic identities. Consider the following:

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2 + e^{-2x}}{2} - \frac{e^{2x} - 2 + e^{-2x}}{2} \Rightarrow \frac{4}{4} \Rightarrow 1$$

And also:

$$2\sinh x \cosh x = 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^{2x} - e^{-2x}}{2} = \sinh 2x$$

Since hyperbolic functions are defined in terms of exponential functions, it is easy to derive rules for their derivatives.

Hyperbolic Identities

$$\cosh^2 x - \sinh^2 x = 1$$

$$\tanh^2 x + \operatorname{sech}^2 x = 1$$

$$\coth^2 x - \operatorname{csch}^2 x = 1$$

$$\sinh 2x = 2\sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}[\sinh x] = \cosh x$$

$$\frac{d}{dx}[\cosh x] = \sinh x$$

$$\frac{d}{dx}[\tanh x] = \operatorname{sech}^2 x$$

$$\frac{d}{dx}[\coth x] = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}[\operatorname{sech} x] = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}[\operatorname{csch} x] = -\operatorname{csch} x \coth x$$

The derivatives of inverse hyperbolic functions resemble.

Inverse Hyperbolic Functions

$$\frac{d}{dx}\left[\sinh^{-1} u\right] = \frac{u'}{\sqrt{u^2 + 1}} \qquad \frac{d}{dx}\left[\operatorname{csch}^{-1} u\right] = \frac{-u'}{|u|\sqrt{1 + u^2}} \qquad \int \frac{du}{\sqrt{a^2 - u^2}} = \ln\left(u + \sqrt{u^2 \pm a^2}\right) + C$$

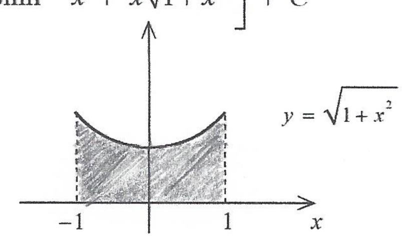
$$\frac{d}{dx}\left[\cosh^{-1} u\right] = \frac{u'}{\sqrt{u^2 - 1}} \qquad \frac{d}{dx}\left[\operatorname{sech}^{-1} u\right] = \frac{-u'}{u\sqrt{1 - u^2}} \qquad \int \frac{du}{\sqrt{u^2 \pm a^2}} = \frac{1}{2a}\ln\left|\frac{a + u}{a - u}\right| + C$$

$$\frac{d}{dx}\left[\tanh^{-1} x\right] = \frac{u'}{1 - u^2} \qquad \frac{d}{dx}\left[\coth^{-1} x\right] = \frac{u'}{1 - u^2} \qquad \int \frac{du}{u\sqrt{a^2 \pm u^2}} = -\frac{1}{a}\ln\frac{a + \sqrt{u^2 \pm a^2}}{|u|} + C$$

Now consider $\int_{-1}^{1} \sqrt{1+x^2}\, dx$. This requires integration by parts $\left(\int u\, dv = uv - \int v\, du\right)$ and hyperbolic substitution.

$$\int \sqrt{1+x^2}\, dx = x\sqrt{1+x^2} - \int \frac{x^2}{\sqrt{1+x^2}}\, dx \Rightarrow x\sqrt{1+x^2} - \frac{1}{2}\left[-\sinh^{-1} x + x\sqrt{1+x^2}\right] + C$$

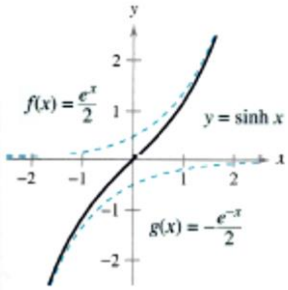
$$\int \sqrt{1+x^2}\, dx = \frac{1}{2}\left[x\sqrt{1+x^2} + \sinh^{-1} x\right] + C$$



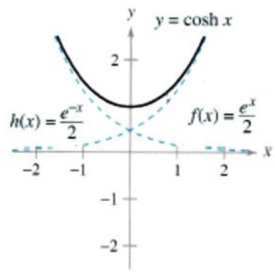
Hence, $\int_{-1}^{1} \sqrt{1+x^2}\, dx = \frac{1}{2}\left[x\sqrt{1+x^2} + \sinh^{-1} x\right]_{-1}^{1} \approx 2.296$

Hyperbolic Functions

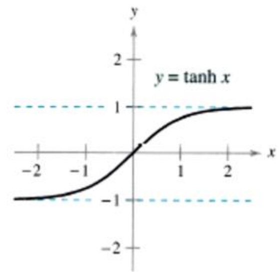
There is a special class of the even and odd combinations of the exponential functions e^x and e^{-x} which occur so frequently they are given a special name: **hyperbolic functions**. They have the same relationship to the hyperbola that trigonometric functions have to the circle.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



Domain: $(-\infty, \infty)$
Range: $[1, \infty)$



Domain: $(-\infty, \infty)$
Range: $(-1, 1)$

$$\sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x} = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x} = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Note that the graphs of $\sinh x$ and $\cosh x$ can be obtained by the addition of ordinates using the exponential functions $y = \frac{1}{2}e^x$ and $y = -\frac{1}{2}e^{-x}$, while $\tanh x$ is obtained from the ratio of $\frac{\sinh x}{\cosh x}$.

Definitions of Hyperbolic Functions

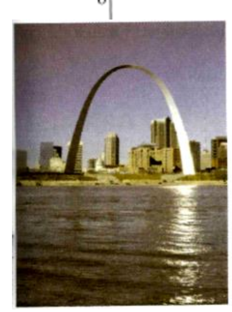
$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

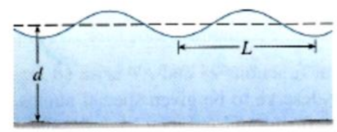
- Some mathematical applications of hyperbolic functions occur in science and engineering when an entity like light, velocity, electricity, or radioactivity is gradually absorbed or extinguished.

- The most common application is the use of hyperbolic cosine to describe the shape of a hanging wire. It can be shown that if a heavy flexible cable is suspended between two points at the same height, it takes the shape of a curve with equation $y = c + a\cosh\left(\frac{x}{a}\right)$, called a *catenary* (from the Latin word *catena*, which means "chain").

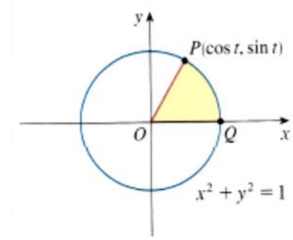


- The Gateway Arch in St. Louis is a structure designed using a hyperbolic cosine function.

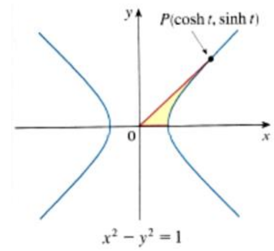
- Another hyperbolic function application occurs in the description of ocean waves. The velocity of a water wave with length L moving across a body of water with depth d is modeled by the function $v = \sqrt{\dfrac{gL}{2\pi}\tanh\left(\dfrac{2\pi d}{L}\right)}$ where g is the acceleration due to gravity.



Recall that if t is any real number, then the point $P(\cos t, \sin t)$ lies on the unit circle $x^2 + y^2 = 1$ because $\cos^2 t + \sin^2 t = 1$, and t can be interpreted as the radian measure of $\angle POQ$, and it represents twice the area of the shaded circular region. This is why trigonometric functions are also called circular functions.



Likewise, if t is any real number, then the point $P(\cosh t, \sinh t)$ lies on the right branch of the hyperbola because $\cosh^2 t - \sinh^2 t = 1$ and $\cosh t \geq 1$. This time, t does not represent the measure of an angle, but it does represent twice the area of the shaded hyperbolic sector.



- Hyperbolic functions provide the ability of using hyperbolic substitutions instead of trigonometric substitutions for radical expressions, and they sometimes lead to simpler answers. However, trigonometric substitutions are more commonly used because trigonometric identities are more familiar than hyperbolic identities.

Consider $\int_{-1}^{1}\sqrt{1-x^2}\,dx$. This requires integration by parts $\left(\int u\,dv = uv - \int v\,du\right)$ and trigonometric substitution.

$$\int\sqrt{1-x^2}\,dx = x\sqrt{1-x^2} - \int\frac{-x^2}{\sqrt{1-x^2}}\,dx$$

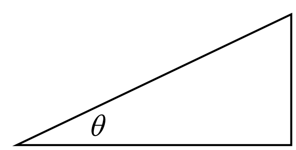
$u = \sqrt{1-x^2}$, $du = \dfrac{-x}{\sqrt{1-x^2}}dx$, $dv = dx$, and $v = x$

$$= x\sqrt{1-x^2} + \int\frac{x^2}{\sqrt{1-x^2}}\,dx$$

$x = \sin\theta, dx = \cos\theta d\theta$, and $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

$$= x\sqrt{1-x^2} + \int\frac{\sin^2\theta}{\cos\theta}(\cos\theta\,d\theta)$$

$$= x\sqrt{1-x^2} + \int\frac{1-\cos 2\theta}{2}\,d\theta$$



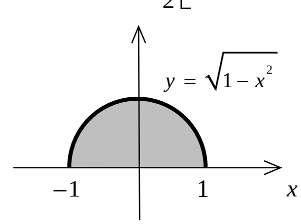
$$= x\sqrt{1-x^2} + \frac{1}{2}\int 1d\theta - \frac{1}{2}\cdot\frac{1}{2}\int\cos 2\theta(2\,d\theta)$$

$$= x\sqrt{1-x^2} + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$

$\theta = \sin^{-1}x$, $\sin 2\theta = 2\sin\theta\cos\theta$

$$= x\sqrt{1-x^2} - \frac{1}{2}\left[\sin^{-1}x - \sin\theta\cos\theta\right] + C \Rightarrow x\sqrt{1-x^2} - \frac{1}{2}\left[\sin^{-1}x - x\sqrt{1-x^2}\right] + C$$

$$\int\sqrt{1-x^2}\,dx = \frac{1}{2}\left[x\sqrt{1-x^2} + \sin^{-1}x\right] + C$$



Hence, $\int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{1}{2}\left[x\sqrt{1-x^2} + \sin^{-1}x\right]_{-1}^{1} = \frac{\pi}{2}$

Many of the trigonometric identities have corresponding hyperbolic identities. Consider the following:

$$\cosh^2 x - \sinh^2 x \;=\; \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \;=\; \frac{e^{2x} + 2 + e^{-2x}}{2} \;-\; \frac{e^{2x} - 2 + e^{-2x}}{2} \;\Rightarrow\; \frac{4}{4} \;\Rightarrow\; 1$$

And also:

$$2\sinh x \cosh x \;=\; 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \;=\; \frac{e^{2x} - e^{-2x}}{2} \;=\; \sinh 2x$$

Since hyperbolic functions are defined in terms of exponential functions, it is easy to derive rules for their derivatives.

Hyperbolic Identities

$$\cosh^2 x - \sinh^2 x = 1$$

$$\tanh^2 x + \operatorname{sech}^2 x = 1$$

$$\coth^2 x - \operatorname{csch}^2 x = 1$$

$$\sinh 2x = 2\sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}\left[\sinh x\right] = \cosh x$$

$$\frac{d}{dx}\left[\cosh x\right] = \sinh x$$

$$\frac{d}{dx}\left[\tanh x\right] = \operatorname{sech}^2 x$$

$$\frac{d}{dx}\left[\coth x\right] = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}\left[\operatorname{sech} x\right] = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}\left[\operatorname{csch} x\right] = -\operatorname{csch} x \coth x$$

The derivatives of inverse hyperbolic functions resemble.

Inverse Hyperbolic Functions

$$\frac{d}{dx}\left[\sinh^{-1} u\right] = \frac{u'}{\sqrt{u^2 + 1}} \qquad \frac{d}{dx}\left[\operatorname{csch}^{-1} u\right] = \frac{-u'}{|u|\sqrt{1 + u^2}} \qquad \int \frac{du}{\sqrt{a^2 - u^2}} = \ln\left(u + \sqrt{u^2 \pm a^2}\right) + C$$

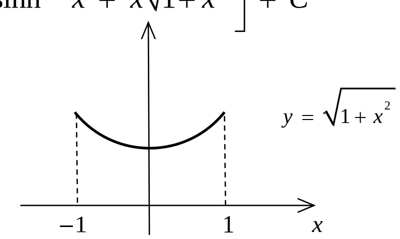
$$\frac{d}{dx}\left[\cosh^{-1} u\right] = \frac{u'}{\sqrt{u^2 - 1}} \qquad \frac{d}{dx}\left[\operatorname{sech}^{-1} u\right] = \frac{-u'}{u\sqrt{1 - u^2}} \qquad \int \frac{du}{\sqrt{u^2 \pm a^2}} = \frac{1}{2a}\ln\left|\frac{a + u}{a - u}\right| + C$$

$$\frac{d}{dx}\left[\tanh^{-1} x\right] = \frac{u'}{1 - u^2} \qquad \frac{d}{dx}\left[\coth^{-1} x\right] = \frac{u'}{1 - u^2} \qquad \int \frac{du}{u\sqrt{a^2 \pm u^2}} = -\frac{1}{a}\ln\frac{a + \sqrt{u^2 \pm a^2}}{|u|} + C$$

Now consider $\int_{-1}^{1}\sqrt{1 + x^2}\,dx$. This requires integration by parts $\left(\int u\,dv = uv - \int v\,du\right)$ and hyperbolic substitution.

$$\int\sqrt{1 + x^2}\,dx \;=\; x\sqrt{1 + x^2} - \int\frac{x^2}{\sqrt{1 + x^2}}\,dx \;\Rightarrow\; x\sqrt{1 + x^2} - \frac{1}{2}\left[-\sinh^{-1} x + x\sqrt{1 + x^2}\right] + C$$

$$\int\sqrt{1 + x^2}\,dx \;=\; \frac{1}{2}\left[x\sqrt{1 + x^2} + \sinh^{-1} x\right] + C$$



Hence, $\int_{-1}^{1}\sqrt{1 + x^2}\,dx \;=\; \frac{1}{2}\left[x\sqrt{1 + x^2} + \sinh^{-1} x\right]_{-1}^{1} \;\approx\; 2.296$

VII

Trigonometric Integrals

- OK, now the integration gets more involved and the commitment to the memorization of trigonometric identities, trig (unit circle) values and trigonometric integration rules up to this point will determine the ease with which students adjust to this new material. The techniques you will now utilize involve integrals which do not conform to the simple *"u du"* or "integration by parts" rules.

- Some identities which you should recall are: (in addition to Circular Function definitions involving *x*, *y* and *r*)

Pythagorean Identities:

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$\cot^2 x + 1 = \csc^2 x$

Product-to-Sum Identities: (2 different angles)

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$

$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$

Double-Angle Identities:

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

Power-Reduction Identities:

$\sin^2 x = \frac{1}{2}(1-\cos 2x) \Rightarrow \frac{1-\cos 2x}{2}$

$\cos^2 x = \frac{1}{2}(1+\cos 2x) \Rightarrow \frac{1+\cos 2x}{2}$

EXAMPLES:

1. $\int \sin 7x\cos 3x\,dx$ *(Use Product-to-Sum Identity)* $\alpha = 7x$ $\beta = 3x$

$\Rightarrow \int \frac{1}{2}(\sin 4x + \sin 10x)\,dx$ $u = 4x$ $v = 10x$

$\Rightarrow \frac{1}{2}\left[\frac{1}{4}\int \sin 4x(4dx) + \frac{1}{10}\int \sin 10x(10dx)\right]$ *Recall:* $\int \sin u\,du = -\cos u + C$

$\Rightarrow -\frac{1}{8}\cos 4x - \frac{1}{20}\cos 10x + C$

2. $\int \sin^4 x\cos^5 x\,dx =$ *(Rewrite the integral, and then use a Pythagorean Identity)*

$\Rightarrow \int \sin^4 x\cos^4 x\cos x\,dx \Rightarrow \int \sin^4 x\left(\cos^2 x\right)^2\cos x\,dx$

$\Rightarrow \int \sin^4 x\left(1-\sin^2 x\right)^2\cos x\,dx$

$\Rightarrow \int \sin^4 x\left(1-2\sin^2 x+\sin^4 x\right)\cos x\,dx$

$\Rightarrow \int \left(\sin^4 x - 2\sin^6 x + \sin^8 x\right)\cos x\,dx$

$\Rightarrow \int \sin^4 x\cos x\,dx - 2\int \sin^6 x\cos x\,dx + \int \sin^8 x\cos x\,dx$ *Note the "u du" form*

$\Rightarrow \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$ *if* $u = \sin x$.

VII

p.2

- By using a similar technique for integrals of similar types, the following rules develop:

$\int \sin^m x \cos^n x\, dx$	***Procedure***	***Relevant Identities***
***n* is odd**	Split off a factor of $\cos x$ Apply the relevant identity Make the *u*-substitution: $u = \sin x$	$\cos^2 x = 1 - \sin^2 x$
***m* is odd**	Split off a factor of $\sin x$ Apply the relevant identity Make the *u*-substitution: $u = \cos x$	$\sin^2 x = 1 - \cos^2 x$
***m* is even**	Reduce the powers on $\sin x$ and $\cos x$	$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ $\cos 2x = \cos^2 x - \sin^2 x = (1 - \sin^2 x) - \sin^2 x \Rightarrow 1 - 2\sin^2 x = \cos 2x$
***n* is even**	(Use relevant identities)	$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ $\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x) \Rightarrow 2\cos^2 x - 1 = \cos 2x$

- Let's look at that last example again, in light of the above rules and using *u*-substitution:

$$\int \sin^4 x \cos^5 x\, dx =$$ ***Use the 1st procedure from above.***

$$\Rightarrow \int \sin^4 x \cos^4 x \cos x\, dx \Rightarrow \int \sin^4 x \left(\cos^2 x\right)^2 \cos x\, dx$$ *Apply Identity*

$$\Rightarrow \int \sin^4 x \left(1 - \sin^2 x\right)^2 \cos x\, dx$$ $u = \sin x, \quad du = \cos x\, dx$

$$\Rightarrow \int u^4 \left(1 - u^2\right)^2 du \Rightarrow \int u^4 \left(1 - 2u^2 + u^4\right) du$$

$$\Rightarrow \int \left(u^4 - 2u^6 + u^8\right) du \Rightarrow \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C$$

$$\Rightarrow \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$$

<u>MORE EXAMPLES</u>:

3. $\int \sin^3 x \cos^4 x\, dx =$ ***Use the 2nd procedure from above.***

$$\Rightarrow \int \sin^2 x \cos^4 x \sin x\, dx \Rightarrow \int \left(1 - \cos^2 x\right) \cos^4 x \sin x\, dx$$ $u = \cos x, \quad du = -\sin x\, dx$

$$\Rightarrow -\int \left(1 - u^2\right) u^4 du \Rightarrow -\int \left(u^4 - u^6\right) du \Rightarrow -\left(\frac{1}{5}u^5 - \frac{1}{7}u^7\right) + C$$

$$\Rightarrow -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$$

VII

4. $\int \sin^4 x \cos^4 x \, dx = \int (\sin^2 x)^2 (\cos^2 x)^2 dx$ *Use the 3rd procedure from above.*

$$\Rightarrow \int \left(\frac{1}{2}[1-\cos 2x]\right)^2 \left(\frac{1}{2}[1+\cos 2x]\right)^2 dx$$

ALOHA

$$\Rightarrow \int \frac{1}{16}(1-\cos 2x)(1-\cos 2x)(1+\cos 2x)(1+\cos 2x)\, dx$$

(DTS pairs) *Pair the conjugate factors!*

$$\Rightarrow \frac{1}{16}\int \left(1-\cos^2 2x\right)^2 dx \Rightarrow \frac{1}{16}\int \left(\sin^2 2x\right)^2 dx \Rightarrow \frac{1}{16}\int \sin^4 2x \, dx \qquad u = 2x, \quad du = 2dx$$

$$\Rightarrow \frac{1}{32}\int \sin^4 u \, du$$

Now, if $v = \sin u$, there is no way to get "dv" so you cannot integrate this easily without combining other identities and integration techniques (Integration by Parts)

Therefore, the following general reduction formulas will be useful:

$$\int \sin^n x \, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \, dx$$

$$\int \cos^n x \, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \, dx$$

These general formulas yield specific cases for the following:

$$\int \sin^2 x \, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C \qquad \int \cos^2 x \, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

$$\int \sin^3 x \, dx = \frac{1}{3}\cos^3 x - \cos x + C \qquad \int \cos^3 x \, dx = \sin x - \frac{1}{3}\sin^3 x + C$$

$$\int \sin^4 x dx = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C \qquad \int \cos^4 x \, dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

Continuing our example, and applying the rule for $\int \sin^4 x \, dx$, the integral $\int \sin^4 u \, du$ becomes:

$$\Rightarrow \frac{1}{32}\left(\frac{3}{8}u - \frac{1}{4}\sin 2u + \frac{1}{32}\sin 4u\right) + C \qquad \textit{Recall: } u = 2x$$

$$\Rightarrow \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C$$

5. Prove the reduction formula for $\int \cos^4 x \, dx$.

$\left(\frac{1}{2} + \frac{\cos 2x}{2}\right)^2$ $\quad \frac{1}{4}\left(\frac{1+\cos 2(2x)}{2}\right)$

$$\int \cos^4 x \, dx \Rightarrow \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \Rightarrow \int \left(\frac{1}{4} + \frac{2\cos 2x}{4} + \frac{\cos^2 2x}{4}\right) dx \Rightarrow \int \left[\frac{1}{4} + \frac{2\cos 2x}{4} + \frac{1}{4}\left(\frac{1+\cos 4x}{2}\right)\right] dx$$

$\int (\cos^2 x)^2 dx$

$$\Rightarrow \frac{1}{4}\int dx + \frac{1}{4}\int \cos 2x(2dx) + \frac{1}{8}\int dx + \frac{1}{32}\int \cos 4x(4dx)$$

$$\Rightarrow \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

VII

6. Prove the reduction formula for $\int \sin^3 x\,dx$.

$$\int \sin^3 x\,dx \Rightarrow \int (\sin^2 x)\sin x\,dx \Rightarrow \int (1-\cos^2 x)\sin x\,dx \qquad u = \cos x,\quad du = -\sin x\,dx$$

$$\Rightarrow -\int (1-u^2)\,du \Rightarrow -\left(u - \frac{1}{3}u^3\right) + C$$

$$\Rightarrow \frac{1}{3}u^3 - u + C \Rightarrow \frac{1}{3}\cos^3 x - \cos x + C$$

*Checking Wallis' Formula

$$\int_0^{\frac{\pi}{2}} \sin^3 x\,dx = \left[\tfrac{1}{3}\cos^3 x - \cos x\right]_0^{\frac{\pi}{2}}$$
$$= \left[\tfrac{1}{3}(0)^3 - 0\right] - \left[\tfrac{1}{3}(1)^3 - 1\right]$$
$$= 0 - \left[\tfrac{1}{3} - 1\right]$$
$$= -\tfrac{1}{3} + 1 = \boxed{\tfrac{2}{3}}$$

♦ ***Likewise, the Tangent and Secant functions also have reduction formulas:***

In general:

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$

$$\int \sec^n x\,dx = \frac{1}{n-1}\left(\sec^{n-2} x \tan x\right) + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

These general formulas yield specific cases for the following:

$$\int \tan^2 x\,dx = \tan x - x + C \qquad \int \sec^2 x\,dx = \tan x + C$$

$$\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x - \ln|\sec x| + C \qquad \int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$ *

Recall: $\int \tan x\,dx = \ln|\sec x| + C$ *and* $\int \sec x\,dx = \ln|\sec x + \tan x| + C$. *

♦ Using integration techniques and substitutions, the following rules develop:

$\int \tan^m x \sec^n x\,dx$	***Procedure***	***Relevant Identities***
***n* is even**	Split off a factor of $\sec^2 x$ Apply the relevant identity Make the u-substitution: $u = \tan x$	$\sec^2 x = \tan^2 x + 1$
***m* is odd**	Split off a factor of $\sec x \tan x$ Apply the relevant identity Make the u-substitution: $u = \sec x$	$\tan^2 x = \sec^2 x - 1$
***m* is even** ***n* is odd**	Reduce the integrand to powers of $\sec x$ alone Use the reduction formula for powers of $\sec x$	$\tan^2 x = \sec^2 x - 1$

VII

MORE EXAMPLES:

7. $\int \tan^2 x \sec^4 x \, dx \Rightarrow$ *Use the 1st procedure from above.*

$$\Rightarrow \int \tan^2 x \sec^2 x \sec^2 x \, dx$$

$$\Rightarrow \int \tan^2 x \left(1+\tan^2 x\right)\sec^2 x \, dx \qquad u = \tan x, \quad du = \sec^2 x \, dx$$

$$\Rightarrow \int u^2\left(u^2+1\right)du$$

$$\Rightarrow \int \left(u^4+u^2\right)du$$

$$\Rightarrow \frac{1}{5}u^5 + \frac{1}{3}u^3 + C \Rightarrow \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C$$

8. $\int \tan^3 x \sec^3 x \, dx \Rightarrow \int \tan^2 x \sec^2 x \left(\sec x \tan x\right)dx$ *Use the 2nd procedure from above.*

$$\Rightarrow \int \left(\sec^2 x - 1\right)\sec^2 x\left(\sec x \tan x\right)dx$$

$$u = \sec x, \quad du = \sec x \tan x \, dx$$

$$\Rightarrow \int \left(u^2-1\right)u^2 du$$

$$\Rightarrow \int \left(u^4-u^2\right)du$$

$$\Rightarrow \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \Rightarrow \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$$

9. $\int \tan^2 x \sec x \, dx \Rightarrow$ $1+\tan^2 x = \sec^2 x$ *Use the 3rd procedure from above.*

$$\Rightarrow \int \left(\sec^2 x - 1\right)\sec x \, dx$$

$$\Rightarrow \int \left(\sec^3 x - \sec x\right)dx$$

$$\Rightarrow \int \sec^3 x \, dx - \int \sec x \, dx$$

$$\Rightarrow \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| - \ln\left|\sec x + \tan x\right| + C$$

$$\Rightarrow \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln\left|\sec x + \tan x\right| + C$$

Wallis' Formulas provide a quick way of evaluating $\int_0^{\frac{\pi}{2}} \cos^n x \, dx$

1. If n is odd $(n \ge 3)$, then

$$\int_0^{\frac{\pi}{2}} \cos^n x \, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$$

2. If n is even $(n \ge 2)$, then

$$\int_0^{\frac{\pi}{2}} \cos^n x \, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

These formulas are also valid if $\cos^n x$ ***is replaced by*** $\sin^n x$.

Check by evaluating Example #6 on $\left[0, \frac{\pi}{2}\right]$. *

$\int_0^{\frac{\pi}{2}} \sin^3 x \, dx$, $n=3$, $= \frac{n-1}{n} = \frac{2}{3}$

Trigonometric Integrals

- OK, now the integration gets more involved and the commitment to the memorization of trigonometric identities, trig (unit circle) values and trigonometric integration rules up to this point will determine the ease with which students adjust to this new material. The techniques you will now utilize involve integrals which do not conform to the simple *"u du"* or "integration by parts" rules.

- Some identities which you should recall are: (in addition to Circular Function definitions involving *x, y* and *r*)

Pythagorean Identities:

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$\cot^2 x + 1 = \csc^2 x$

Product-to-Sum Identities: (2 different angles)

$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$

$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$

$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$

Double-Angle Identities:

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x \;=\; 2\cos^2 x - 1 \;=\; 1 - 2\sin^2 x$

Power-Reduction Identities:

$\sin^2 x = \frac{1}{2}\left(1-\cos 2x\right)$

$\cos^2 x = \frac{1}{2}\left(1+\cos 2x\right)$

EXAMPLES:

1. $\int \sin 7x\cos 3x\,dx$ *(Use Product-to-Sum Identity)* $\alpha = 7x$ $\beta = 3x$

$\Rightarrow \int \frac{1}{2}\left(\sin 4x + \sin 10x\right)dx$ $u = 4x$ $v = 10x$

$\Rightarrow \frac{1}{2}\left[\frac{1}{4}\int \sin 4x(4dx) + \frac{1}{10}\int \sin 10x(10dx)\right]$ *Recall:* $\int \sin u\,du = -\cos u + C$

$\Rightarrow -\frac{1}{8}\cos 4x - \frac{1}{20}\cos 10x + C$

2. $\int \sin^4 x\cos^5 x\,dx =$ *(Rewrite the integral, and then use a Pythagorean Identity)*

$\Rightarrow \int \sin^4 x\cos^4 x\cos x\,dx \Rightarrow \int \sin^4 x\left(\cos^2 x\right)^2\cos x\,dx$

$\Rightarrow \int \sin^4 x\left(1-\sin^2 x\right)^2\cos x\,dx$

$\Rightarrow \int \sin^4 x\left(1-2\sin^2 x+\sin^4 x\right)\cos x\,dx$

$\Rightarrow \int\left(\sin^4 x-2\sin^6 x+\sin^8 x\right)\cos x\,dx$

$\Rightarrow \int \sin^4 x\cos x\,dx - 2\int \sin^6 x\cos x\,dx + \int \sin^8 x\cos x\,dx$ *Note the "u du" form*

$\Rightarrow \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$ *if* $u = \sin x$.

♦ By using a similar technique for integrals of similar types, the following rules develop:

$\int \sin^m x \cos^n x\, dx$	***Procedure***	***Relevant Identities***
***n* is odd**	Split off a factor of $\cos x$ Apply the relevant identity Make the *u*-substitution: $u = \sin x$	$\cos^2 x = 1 - \sin^2 x$
***m* is odd**	Split off a factor of $\sin x$ Apply the relevant identity Make the *u*-substitution: $u = \cos x$	$\sin^2 x = 1 - \cos^2 x$
***m* is even**	Reduce the powers on $\sin x$ and $\cos x$	$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
***n* is even**	(Use relevant identities)	$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

♦ Let's look at that last example again, in light of the above rules and using *u*-substitution:

$$\int \sin^4 x \cos^5 x\, dx =$$ ***Use the 1st procedure from above.***

$$\Rightarrow \int \sin^4 x \cos^4 x \cos x\, dx \Rightarrow \int \sin^4 x \left(\cos^2 x\right)^2 \cos x\, dx$$ *Apply Identity*

$$\Rightarrow \int \sin^4 x \left(1 - \sin^2 x\right)^2 \cos x\, dx$$ $u = \sin x, \quad du = \cos x\, dx$

$$\Rightarrow \int u^4 \left(1 - u^2\right)^2 du \Rightarrow \int u^4 \left(1 - 2u^2 + u^4\right) du$$

$$\Rightarrow \int \left(u^4 - 2u^6 + u^8\right) du \Rightarrow \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C$$

$$\Rightarrow \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$$

<u>MORE EXAMPLES</u>:

3. $\int \sin^3 x \cos^4 x\, dx =$ ***Use the 2nd procedure from above.***

$$\Rightarrow \int \sin^2 x \cos^4 x \sin x\, dx \Rightarrow \int \left(1 - \cos^2 x\right) \cos^4 x \sin x\, dx$$ $u = \cos x, \quad du = -\sin x\, dx$

$$\Rightarrow -\int \left(1 - u^2\right) u^4 du \Rightarrow -\int \left(u^4 - u^6\right) du \Rightarrow -\left(\frac{1}{5}u^5 - \frac{1}{7}u^7\right) + C$$

$$\Rightarrow -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$$

4. $\int \sin^4 x \cos^4 x \, dx \ =$ ***Use the 3rd procedure from above.***

$$\Rightarrow \int \left(\frac{1}{2}[1-\cos 2x]\right)^2 \left(\frac{1}{2}[1+\cos 2x]\right)^2 dx$$

$$\Rightarrow \int \frac{1}{16}(1-\cos 2x)(1-\cos 2x)(1+\cos 2x)(1+\cos 2x)\,dx$$ *Pair the conjugate factors!*

$$\Rightarrow \frac{1}{16}\int \left(1-\cos^2 2x\right)^2 dx \ \Rightarrow \ \frac{1}{16}\int \left(\sin^2 2x\right)^2 dx \ \Rightarrow \ \frac{1}{16}\int \sin^4 2x \, dx$$ $u = 2x, \quad du = 2dx$

$$\Rightarrow \frac{1}{32}\int \sin^4 u \, du$$ *Now, if* $v = \sin u$*, there is no way to get "dv" so you cannot integrate this easily without combining other identities and integration techniques (Integration by Parts)*

Therefore, the following general reduction formulas will be useful:

$$\int \sin^n x \, dx \ = \ -\frac{1}{n}\sin^{n-1} x \cos x \ + \ \frac{n-1}{n}\int \sin^{n-2} x \, dx$$

$$\int \cos^n x \, dx \ = \ \frac{1}{n}\cos^{n-1} x \sin x \ + \ \frac{n-1}{n}\int \cos^{n-2} x \, dx$$

These general formulas yield specific cases for the following:

$$\int \sin^2 x \, dx \ = \ \frac{1}{2}x \ - \ \frac{1}{4}\sin 2x \ + \ C \qquad \int \cos^2 x \, dx \ = \ \frac{1}{2}x \ + \ \frac{1}{4}\sin 2x \ + \ C$$

$$\int \sin^3 x \, dx \ = \ \frac{1}{3}\cos^3 x \ - \ \cos x \ + \ C \qquad \int \cos^3 x \, dx \ = \ \sin x \ - \ \frac{1}{3}\sin^3 x \ + \ C$$

$$\int \sin^4 x dx \ = \ \frac{3}{8}x \ - \ \frac{1}{4}\sin 2x \ + \ \frac{1}{32}\sin 4x \ + \ C \qquad \int \cos^4 x \, dx \ = \ \frac{3}{8}x \ + \ \frac{1}{4}\sin 2x \ + \ \frac{1}{32}\sin 4x \ + \ C$$

Continuing our example, and applying the rule for $\int \sin^4 x \, dx$*, the integral* $\int \sin^4 u \, du$ *becomes:*

$$\Rightarrow \frac{1}{32}\left(\frac{3}{8}u \ - \ \frac{1}{4}\sin 2u \ + \ \frac{1}{32}\sin 4u\right) \ + \ C$$ *Recall:* $u = 2x$

$$\Rightarrow \frac{3}{128}x \ - \ \frac{1}{128}\sin 4x \ + \ \frac{1}{1024}\sin 8x \ + \ C$$

5. Prove the reduction formula for $\int \cos^4 x \, dx$.

$$\int \cos^4 x \, dx \ \Rightarrow \ \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \ \Rightarrow \ \int \left(\frac{1}{4} + \frac{2\cos 2x}{4} + \frac{\cos^2 2x}{4}\right) dx \ \Rightarrow \ \int \left[\frac{1}{4} + \frac{2\cos 2x}{4} + \frac{1}{4}\left(\frac{1+\cos 4x}{2}\right)\right] dx$$

$$\Rightarrow \frac{1}{4}\int dx + \frac{1}{4}\int \cos 2x(2dx) + \frac{1}{8}\int dx + \frac{1}{32}\int \cos 4x(4dx)$$

$$\Rightarrow \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

6. Prove the reduction formula for $\int \sin^3 x\, dx$.

$$\int \sin^3 x\, dx \;\Rightarrow\; \int \left(\sin^2 x\right)\sin x\, dx \;\Rightarrow\; \int \left(1-\cos^2 x\right)\sin x\, dx \qquad u=\cos x,\;\; du=-\sin x\, dx$$

$$\Rightarrow\; -\int \left(1-u^2\right) du \;\Rightarrow\; -\left(u-\frac{1}{3}u^3\right) + C$$

$$\Rightarrow\; \frac{1}{3}u^3 - u + C \;\Rightarrow\; \frac{1}{3}\cos^3 x - \cos x + C$$

- ***Likewise, the Tangent and Secant functions also have reduction formulas:***

In general:

$$\int \tan^n x\, dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\, dx$$

$$\int \sec^n x\, dx = \frac{1}{n-1}\left(\sec^{n-2} x \tan x\right) + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx$$

These general formulas yield specific cases for the following:

$$\int \tan^2 x\, dx = \tan x - x + C \qquad \int \sec^2 x\, dx = \tan x + C$$

$$\int \tan^3 x\, dx = \frac{1}{2}\tan^2 x - \ln\left|\sec x\right| + C \qquad \int \sec^3 x dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + C$$

Recall: $\int \tan x\, dx = \ln\left|\sec x\right| + C$ *and* $\int \sec x\, dx = \ln\left|\sec x + \tan x\right| + C$.

- Using integration techniques and substitutions, the following rules develop:

$\int \tan^m x \sec^n x\, dx$	***Procedure***	***Relevant Identities***
***n* is even**	Split off a factor of $\sec^2 x$ Apply the relevant identity Make the *u*-substitution: $u = \tan x$	$\sec^2 x = \tan^2 x + 1$
***m* is odd**	Split off a factor of $\sec x \tan x$ Apply the relevant identity Make the *u*-substitution: $u = \sec x$	$\tan^2 x = \sec^2 x - 1$
***m* is even** ***n* is odd**	Reduce the integrand to powers of $\sec x$ alone Use the reduction formula for powers of $\sec x$	$\tan^2 x = \sec^2 x - 1$

MORE EXAMPLES:

7. $\int \tan^2 x \sec^4 x \, dx \Rightarrow$ ***Use the 1st procedure from above.***

$$\Rightarrow \int \tan^2 x \sec^2 x \sec^2 x \, dx$$

$$\Rightarrow \int \tan^2 x \left(1+\tan^2 x\right)\sec^2 x \, dx \qquad u = \tan x, \quad du = \sec^2 x \, dx$$

$$\Rightarrow \int u^2\left(u^2+1\right)du$$

$$\Rightarrow \int \left(u^4+u^2\right)du$$

$$\Rightarrow \frac{1}{5}u^5 + \frac{1}{3}u^3 + C \Rightarrow \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C$$

8. $\int \tan^3 x \sec^3 x \, dx \Rightarrow \int \tan^2 x \sec^2 x \left(\sec x \tan x\right)dx$ ***Use the 2nd procedure from above.***

$$\Rightarrow \int \left(\sec^2 x - 1\right)\sec^2 x\left(\sec x \tan x\right)dx$$

$$u = \sec x, \quad du = \sec x \tan x \, dx$$

$$\Rightarrow \int \left(u^2-1\right)u^2 du$$

$$\Rightarrow \int \left(u^4-u^2\right)du$$

$$\Rightarrow \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \Rightarrow \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$$

9. $\int \tan^2 x \sec x \, dx \Rightarrow$ ***Use the 3rd procedure from above.***

$$\Rightarrow \int \left(\sec^2 x - 1\right)\sec x \, dx$$

$$\Rightarrow \int \left(\sec^3 x - \sec x\right)dx$$

$$\Rightarrow \int \sec^3 x \, dx - \int \sec x \, dx$$

$$\Rightarrow \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| - \ln\left|\sec x + \tan x\right| + C$$

$$\Rightarrow \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln\left|\sec x + \tan x\right| + C$$

Wallis' Formulas provide a quick way of evaluating $\int_0^{\frac{\pi}{2}} \cos^n x \, dx$

1. If n is odd $(n \geq 3)$, then

$$\int_0^{\frac{\pi}{2}} \cos^n x \, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$$

2. If n is even $(n \geq 2)$, then

$$\int_0^{\frac{\pi}{2}} \cos^n x \, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

These formulas are also valid if $\cos^n x$ ***is replaced by*** $\sin^n x$.

Check by evaluating Example #6 on $\left[0, \frac{\pi}{2}\right]$.

Trigonometric Substitution

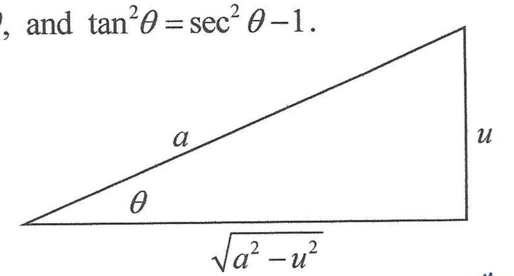
- We are concerned with integrals for which none of the basic rules apply and which contain expressions of the form:

$$\sqrt{a^2-u^2}, \quad \sqrt{a^2+u^2}, \quad \text{and} \quad \sqrt{u^2-a^2}$$

in which a is a positive constant and u is a function of x. The basic idea for evaluating such integrals is to make a substitution for x that will eliminate the radical in the integrand. We will do this with the Pythagorean Identities for $\cos^2\theta = 1-\sin^2\theta$, $\sec^2\theta = 1+\tan^2\theta$, and $\tan^2\theta = \sec^2\theta - 1$.

Consider the right triangle:

$$\sin\theta = \frac{u}{a} \quad \Rightarrow \quad u = a\sin\theta.$$



sin θ = u/a

$$\sqrt{a^2-u^2} \Rightarrow \sqrt{a^2-a^2\sin^2\theta}$$
$$\Rightarrow \sqrt{a^2(1-\sin^2\theta)}$$
$$\Rightarrow \sqrt{a^2\cos^2\theta} \Rightarrow a\cos\theta$$

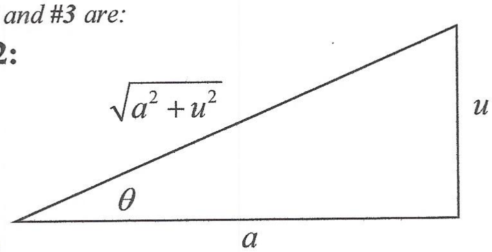
cos θ > 0 since θ is acute

- The following relationships arise:

	Expression in Integrand	*Substitution*	*Restriction on θ*	*Simplification*
1.	$\sqrt{a^2-u^2}$	$u = a\sin\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$a^2-u^2 = a^2-a^2\sin^2\theta = a^2\cos^2\theta$
2.	$\sqrt{a^2+u^2}$	$u = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$a^2+u^2 = a^2+a^2\tan^2\theta = a^2\sec^2\theta$
3.	$\sqrt{u^2-a^2}$	$u = a\sec\theta$	$\begin{cases} 0 \le \theta < \frac{\pi}{2}, & \text{(if } u \ge a) \\ \frac{\pi}{2} < \theta \le \pi, & \text{(if } u \le -a) \end{cases}$	$u^2-a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

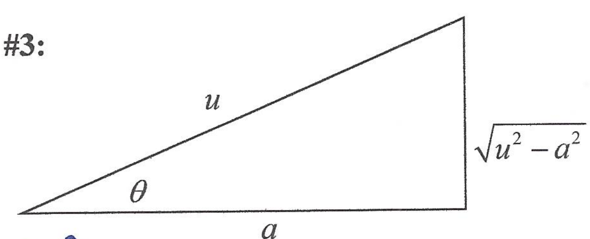
The triangles for #2 and #3 are:

#2:



tan θ = u/a

#3:



sec θ = a/u

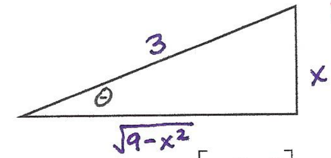
- Note that it is important to be able to correctly set up the triangle as you solve these problems.

- There is a new twist to changing the limits of integration. You must be able to express the upper and lower limits as an <u>angle</u> so you can evaluate the integral.

VIII

p.2

1. Evaluate: $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$ $\qquad x = 3\sin\theta$ and $dx = 3\cos\theta\, d\theta$



Also, $\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3\cos\theta$ $\qquad$ *Note:* $\cos\theta > 0$ *on* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{3\cos\theta}{9\sin^2\theta}\,3\cos\theta\, d\theta$$

$1 + \cot^2\theta = \csc^2\theta$

$$\Rightarrow \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta \Rightarrow \int \cot^2\theta\, d\theta \Rightarrow \int \left(\csc^2\theta - 1\right) d\theta$$

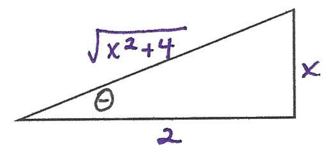
$$= -\cot\theta - \theta + C$$

$x = 3\sin\theta$
$\frac{x}{3} = \sin\theta$
$\theta = \sin^{-1}\left(\frac{x}{3}\right)$

And since $\cot\theta = \frac{\sqrt{9-x^2}}{x}$

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

2. Evaluate $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$ $\qquad x = 2\tan\theta$ and $dx = 2\sec\theta\, d\theta$



Also, $\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4} = \sqrt{4\left(\tan^2\theta + 1\right)} = \sqrt{4\sec^2\theta} = 2\sec\theta$ $\qquad$ *Note:* θ *on* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta \cdot 2\sec\theta}$$

$$\Rightarrow \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$ *Put everything in terms of sine and cosine*

$$\Rightarrow \frac{1}{4}\int \left(\frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta}\right) d\theta = \frac{1}{4}\int \left(\frac{\cos\theta}{\sin^2\theta}\right) d\theta$$ $\qquad u = \sin\theta$ and $du = \cos\theta\, d\theta$

$$\Rightarrow \frac{1}{4}\int u^{-2}\,du = -\frac{1}{4}u^{-1} + C \Rightarrow -\frac{1}{4\sin\theta} + C$$

$$\Rightarrow -\frac{\csc\theta}{4} + C$$ $\qquad$ *And since* $\csc\theta = \frac{\sqrt{x^2+4}}{x}$

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = -\frac{\sqrt{x^2+4}}{4x} + C$$

Some Special Integration Formulas:

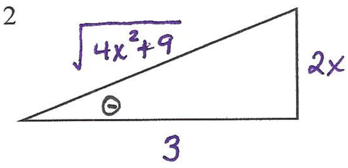
$$\int \sqrt{a^2 - u^2}\,du = \frac{1}{2}\left(a^2 \arcsin\frac{u}{a} + u\sqrt{a^2-u^2}\right) + C$$

$$\int \sqrt{u^2 - a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2-a^2} - a^2\ln\left|u + \sqrt{u^2-a^2}\right|\right) + C, \quad u > a$$

$$\int \sqrt{u^2 + a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2+a^2} + a^2\ln\left|u + \sqrt{u^2+a^2}\right|\right) + C$$

3. Evaluate $\int_0^{3\sqrt{3}/2} \frac{x^3}{\left(4x^2+9\right)^{3/2}}\,dx$

$u = 2x,\quad x = \frac{3}{2}\tan\theta$ and $dx = \frac{3}{2}\sec^2\theta\, d\theta$



Also $\sqrt{4x^2+9} = \sqrt{9\tan^2\theta + 9} = 3\sec\theta$

When $x = 0$, $\tan\theta = 0$, so $\theta = 0$. When $x = 3\sqrt{3}/2$, $\tan\theta = \sqrt{3}$, so $\theta = \pi/3$.

$\tan\theta = \frac{2x}{3}$

$\tan\theta = \frac{2}{3}\left(\frac{3\sqrt{3}}{2}\right)$

$\tan\theta = \sqrt{3}$

$\theta = \frac{\pi}{3}$

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{\left(4x^2+9\right)^{3/2}}\,dx = \int_0^{\pi/3} \frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta}\,\frac{3}{2}\sec^2\theta\, d\theta$$

$$\Rightarrow \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\,d\theta = \frac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta}\,d\theta$$

$$\Rightarrow \frac{3}{16}\int_0^{\pi/3} \frac{1-\cos^2\theta}{\cos^2\theta}\sin\theta\, d\theta$$

$u = \cos\theta$ and $du = -\sin\theta\, d\theta$

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{\left(4x^2+9\right)^{3/2}}\,dx = -\frac{3}{16}\int_1^{1/2} \frac{1-u^2}{u^2}\,du$$

When $\theta = 0$, $u = 1$. When $\theta = 3$, $u = \frac{1}{2}$

$$= \frac{3}{16}\int_1^{1/2}\left(1 - u^{-2}\right)du = \frac{3}{16}\left(u + \frac{1}{u}\right]_1^{1/2}$$

$$= \frac{3}{16}\left[\left(\frac{1}{2}+2\right) - (1+1)\right] = \frac{3}{32}$$

4. Find the arc length of the graph of $f(x) = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$.

$$s = \int_0^1 \sqrt{1 + \left[f'(x)\right]^2}\,dx$$

Recall formula for arc length

$$= \int_0^1 \sqrt{1+x^2}\,dx$$

$f'(x) = x$

$$= \int_0^{\pi/4} \sec^3\theta\, d\theta$$

$a = 1,\ x = \tan\theta$, and $dx = \sec^2\theta\, d\theta$

Recall: $\int \sec^3 x dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + C$

$$= \left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right|\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\sqrt{2} + \ln\left(\sqrt{2}+1\right)\right] \approx 1.148$$

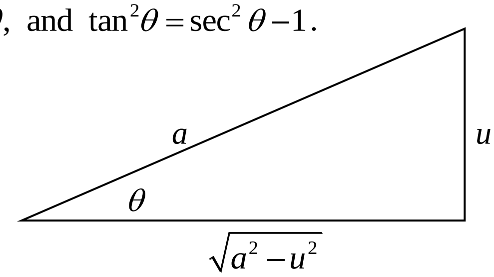
Trigonometric Substitution

- We are concerned with integrals for which none of the basic rules apply and which contain expressions of the form:

$$\sqrt{a^2-u^2}, \quad \sqrt{a^2+u^2}, \quad \text{and} \quad \sqrt{u^2-a^2}$$

in which *a* is a positive constant and *u* is a function of *x*. The basic idea for evaluating such integrals is to make a substitution for *x* that will eliminate the radical in the integrand. We will do this with the Pythagorean Identities for $\cos^2\theta = 1-\sin^2\theta$, $\sec^2\theta = 1+\tan^2\theta$, and $\tan^2\theta = \sec^2\theta - 1$.
Consider the right triangle:

$$\sin\theta = \frac{u}{a} \quad \Rightarrow \quad u = a\sin\theta .$$



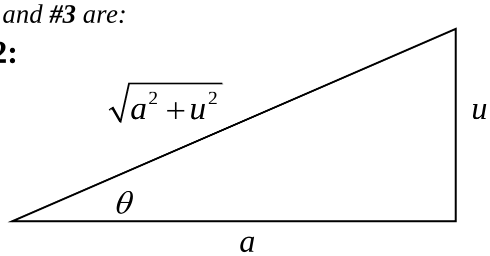
$$\sqrt{a^2-u^2} \;\Rightarrow\; \sqrt{a^2-a^2\sin^2\theta}$$
$$\Rightarrow\; \sqrt{a^2(1-\sin^2\theta)}$$
$$\Rightarrow\; \sqrt{a^2\cos^2\theta} \;\Rightarrow\; a\cos\theta$$

- The following relationships arise:

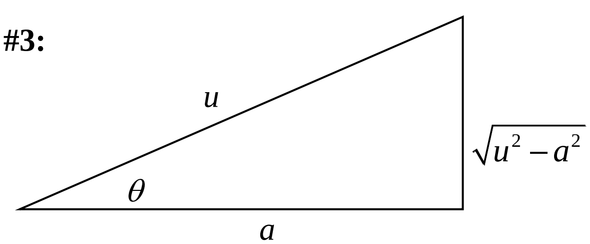
	Expression in Integrand	***Substitution***	***Restriction on*** θ	***Simplification***
1.	$\sqrt{a^2-u^2}$	$u = a\sin\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$a^2-u^2 = a^2-a^2\sin^2\theta = a^2\cos^2\theta$
2.	$\sqrt{a^2+u^2}$	$u = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$a^2+u^2 = a^2+a^2\tan^2\theta = a^2\sec^2\theta$
3.	$\sqrt{u^2-a^2}$	$u = a\sec\theta$	$\begin{cases} 0 \le \theta < \frac{\pi}{2}, & (\text{if } u \ge a) \\ \frac{\pi}{2} < \theta \le \pi, & (\text{if } u \le -a) \end{cases}$	$u^2-a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

*The triangles for **#2** and **#3** are:*

#2:



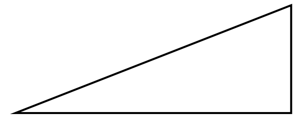
#3:



- Note that it is important to be able to correctly set up the triangle as you solve these problems.

- There is a new twist to changing the limits of integration. You must be able to express the upper and lower limits as an <u>angle</u> so you can evaluate the integral.

1. Evaluate: $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$ $\qquad x = 3\sin\theta$ and $dx = 3\cos\theta\, d\theta$



Also, $\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3\cos\theta$ $\qquad$ *Note:* $\cos\theta > 0$ *on* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{3\cos\theta}{9\sin^2\theta}\, 3\cos\theta\, d\theta$$

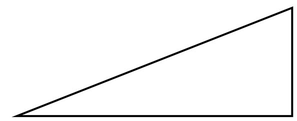
$$\Rightarrow \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta \Rightarrow \int \cot^2\theta\, d\theta \Rightarrow \int \left(\csc^2\theta - 1\right) d\theta$$

$$= -\cot\theta - \theta + C$$

And since $\cot\theta = \frac{\sqrt{9-x^2}}{x}$

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

2. Evaluate $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$ $\qquad x = 2\tan\theta$ and $dx = 2\sec\theta\, d\theta$



Also, $\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4} = \sqrt{4\left(\tan^2\theta + 1\right)} = \sqrt{4\sec^2\theta} = 2\sec\theta$ $\qquad$ *Note:* θ *on* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta \cdot 2\sec\theta}$$

$$\Rightarrow \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

Put everything in terms of sine and cosine

$$\Rightarrow \frac{1}{4}\int \left(\frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta}\right) d\theta = \frac{1}{4}\int \left(\frac{\cos\theta}{\sin^2\theta}\right) d\theta$$

$u = \sin\theta$ and $du = \cos\theta\, d\theta$

$$\Rightarrow \frac{1}{4}\int u^{-2}\,du = -\frac{1}{4}u^{-1} + C \Rightarrow -\frac{1}{4\sin\theta} + C$$

$$\Rightarrow -\frac{\csc\theta}{4} + C$$

And since $\csc\theta = \frac{\sqrt{x^2+4}}{x}$

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = -\frac{\sqrt{x^2+4}}{4x} + C$$

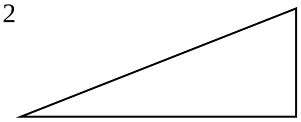
Some Special Integration Formulas:

$$\int \sqrt{a^2-u^2}\,du = \frac{1}{2}\left(a^2 \arcsin\frac{u}{a} + u\sqrt{a^2-u^2}\right) + C$$

$$\int \sqrt{u^2-a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2-a^2} - a^2 \ln\left|u + \sqrt{u^2-a^2}\right|\right) + C, \quad u > a$$

$$\int \sqrt{u^2+a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2+a^2} + a^2 \ln\left|u + \sqrt{u^2+a^2}\right|\right) + C$$

3. Evaluate $\int_0^{3\sqrt{3}/2} \frac{x^3}{\left(4x^2+9\right)^{3/2}}\, dx$ $u = 2x, \quad x = \frac{3}{2}\tan\theta \quad \text{and} \quad dx = \frac{3}{2}\sec^2\theta\, d\theta$



Also $\sqrt{4x^2 + 9} \; = \; \sqrt{9\tan^2\theta + 9} \; = \; 3\sec\theta$

When $x = 0$, $\tan\theta = 0$, so $\theta = 0$. When $x = 3\sqrt{3}/2$, $\tan\theta = \sqrt{3}$, so $\theta = \pi/3$.

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{\left(4x^2+9\right)^{3/2}}\, dx \; = \; \int_0^{\pi/3} \frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta}\, \frac{3}{2}\sec^2\theta\, d\theta$$

$$\Rightarrow \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\, d\theta \; = \; \frac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta}\, d\theta$$

$$\Rightarrow \frac{3}{16}\int_0^{\pi/3} \frac{1-\cos^2\theta}{\cos^2\theta}\, \sin\theta\, d\theta$$ $u = \cos\theta$ and $du = -\sin\theta\, d\theta$

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{\left(4x^2+9\right)^{3/2}}\, dx \; = \; -\frac{3}{16}\int_1^{1/2} \frac{1-u^2}{u^2}\, du$$ When $\theta = 0$, $u = 1$. When $\theta = 3$, $u = \frac{1}{2}$

$$= \; \frac{3}{16}\int_1^{1/2} \left(1 - u^{-2}\right) du \; = \; \frac{3}{16}\left(u + \frac{1}{u}\right]_1^{1/2}$$

$$= \; \frac{3}{16}\left[\left(\frac{1}{2} + 2\right) - \left(1 + 1\right)\right] \; = \; \frac{3}{32}$$

4. Find the arc length of the graph of $f(x) = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$.

$$s \; = \; \int_0^1 \sqrt{1 + \left[f'(x)\right]^2}\, dx$$ *Recall formula for arc length*

$$= \int_0^1 \sqrt{1 + x^2}\, dx$$ $f'(x) = x$

$$= \int_0^{\pi/4} \sec^3\theta\, d\theta$$ $a = 1, \; x = \tan\theta, \; \text{and} \; dx = \sec^2\theta\, d\theta$

Recall: $\int \sec^3 x dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + C$

$$= \left[\frac{1}{2}\sec\theta\tan\theta \; + \; \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right|\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\sqrt{2} + \ln\left(\sqrt{2} + 1\right)\right] \approx 1.148$$

IX

Double Integrals and Volume

Find the volume of the solid region bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the xy-plane.

$z = 0$ in the xy-plane, so the base region is the ellipse $x^2 + 2y^2 = 4$

Variable bounds for y: $-\sqrt{\dfrac{4-x^2}{2}} \le y \le \sqrt{\dfrac{4-x^2}{2}}$

Constant bounds for x: $-2 \le x \le 2$

$$V = \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \left(4 - x^2 - 2y^2\right) dy\, dx$$

$$= \int_{-2}^{2}\left[\left(4 - x^2\right)y - \frac{2y^3}{3}\right]_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} dx$$

ALOHA!

$$= \int_{-2}^{2}\left[\left((4-x^2)\sqrt{\frac{4-x^2}{2}} - \frac{2\left(\sqrt{\frac{4-x^2}{2}}\right)^3}{3}\right) - \left(-(4-x^2)\sqrt{\frac{4-x^2}{2}} - \left(-\frac{2\left(\sqrt{\frac{4-x^2}{2}}\right)^3}{3}\right)\right)\right] dx$$

$$= \int_{-2}^{2} \frac{(4-x^2)(4-x^2)^{1/2}}{\sqrt{2}} - \frac{2}{3}\cdot\frac{(4-x^2)(4-x^2)^{1/2}}{2\sqrt{2}} + \frac{(4-x^2)(4-x^2)^{1/2}}{\sqrt{2}} - \frac{2(4-x^2)(4-x^2)^{1/2}}{3\cdot 2\sqrt{2}}$$

FROF

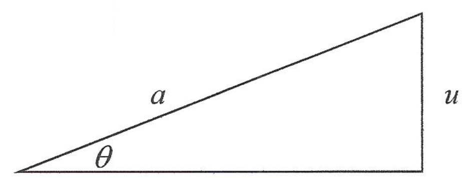
$$= \int_{-2}^{2} \frac{2(4-x^2)(4-x^2)^{1/2}}{\sqrt{2}} - \frac{4\,(4-x^2)(4-x^2)^{1/2}}{3\cdot 2\sqrt{2}}$$

$$\int_{-2}^{2} \frac{2(4-x^2)^{3/2}}{\sqrt{2}} - \frac{2(4-x^2)^{3/2}}{3\sqrt{2}}\, dx$$

$$\int_{-2}^{2} \frac{2}{\sqrt{2}}(4-x^2)^{3/2}\left(1 - \frac{1}{3}\right) dx \qquad \frac{2}{3}$$

$$\int_{-2}^{2} \frac{4}{3\sqrt{2}}(4-x^2)\, dx$$

$$= \frac{4}{3\sqrt{2}}\int_{-2}^{2}\left(4 - x^2\right)^{3/2} dx \Rightarrow \frac{4}{3\sqrt{2}}\int_{-2}^{2}\left[\left(4 - x^2\right)^{1/2}\right]^3 dx$$



$\sin\theta = \dfrac{x}{a} \Rightarrow x = a\sin\theta \qquad \sqrt{a^2 - u^2}$

$dx = a\cos\theta\, d\theta$

$\sqrt{a^2 - u^2} \Rightarrow \sqrt{a^2 - a^2\sin^2\theta} \Rightarrow \sqrt{a^2(1 - \sin^2\theta)}$

$\Rightarrow \sqrt{a^2\cos^2\theta} \Rightarrow a\cos\theta$

$\sqrt{4-x^2} = \sqrt{4 - 4\sin^2\theta} = \sqrt{4(1-\sin^2\theta)} = 2\sqrt{\cos^2\theta} = 2\cos\theta$

$\sqrt{4\cos^2\theta} = 2\cos\theta$

If $x = 2\sin\theta$, *then* $2\sin\theta = \pm 2 \Rightarrow \sin\theta = \pm 1$, *and* $\theta = \pm\dfrac{\pi}{2}$

so $\int_{-\pi/2}^{\pi/2} (2\cos\theta)^3 (2\cos\theta\, d\theta)$

$$= \frac{4}{3\sqrt{2}}\int_{-\pi/2}^{\pi/2} (2\cos\theta)^3 (2\cos\theta\, d\theta)$$

$$= \frac{4}{3\sqrt{2}}\int_{-\pi/2}^{\pi/2} 16\,(\cos\theta)^4\, d\theta \Rightarrow \frac{64}{3\sqrt{2}}(2)\int_{0}^{\pi/2} (\cos\theta)^4\, d\theta$$

Use Wallis' Formula: $\displaystyle\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$

$$= \frac{128}{3\sqrt{2}}\left(\frac{3\pi}{16}\right) \Rightarrow 4\sqrt{2}\pi$$

Double Integrals and Volume

Find the volume of the solid region bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the *xy*-plane.

$z = 0$ in the *xy*-plane, so the base region is the ellipse $x^2 + 2y^2 = 4$

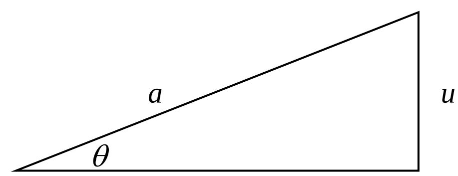
Variable bounds for *y*: $-\sqrt{\dfrac{4-x^2}{2}} \le y \le \sqrt{\dfrac{4-x^2}{2}}$

Constant bounds for *x*: $-2 \le x \le 2$

$$V = \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \left(4 - x^2 - 2y^2\right) dy\, dx$$

$$= \int_{-2}^{2}\left[\left(4 - x^2\right)y - \frac{2y^3}{3}\right]_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} dx$$

$$= \frac{4}{3\sqrt{2}}\int_{-2}^{2}\left(4 - x^2\right)^{3/2} dx \Rightarrow \frac{4}{3\sqrt{2}}\int_{-2}^{2}\left[\left(4 - x^2\right)^{1/2}\right]^3 dx$$



$\sin\theta = \dfrac{x}{a} \Rightarrow x = a\sin\theta$ $\qquad \sqrt{a^2 - u^2}$

$dx = a\cos\theta\, d\theta$

$\sqrt{a^2 - u^2} \Rightarrow \sqrt{a^2 - a^2\sin^2\theta} \Rightarrow \sqrt{a^2(1 - \sin^2\theta)}$

$\Rightarrow \sqrt{a^2\cos^2\theta} \Rightarrow a\cos\theta$

If $x = 2\sin\theta$, *then* $2\sin\theta = \pm 2 \Rightarrow \sin\theta = \pm 1$, *and* $\theta = \pm\dfrac{\pi}{2}$

$$= \frac{4}{3\sqrt{2}}\int_{-\pi/2}^{\pi/2}\left(2\cos\theta\right)^3\left(2\cos\theta\, d\theta\right)$$

$$= \frac{4}{3\sqrt{2}}\int_{-\pi/2}^{\pi/2} 16\left(\cos\theta\right)^4 d\theta \Rightarrow \frac{64}{3\sqrt{2}}(2)\int_{0}^{\pi/2}\left(\cos\theta\right)^4 d\theta$$

Use Wallis' Formula: $\displaystyle\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$

$$= \frac{128}{3\sqrt{2}}\left(\frac{3\pi}{16}\right) \Rightarrow 4\sqrt{2}\pi$$

X

Cobb-Douglas Production Models

(Why do we care about implicit differentiation?)

In economics, a production model is a mathematical relationship between the output of a company or a country and the labor and capital equipment required to produce that output. Much of the pioneering work in the field of production models occurred in the 1920s when Paul Douglas of the University of Chicago and his collaborator Charles Cobb proposed that the output P can be expressed in terms of the labor L and the capital equipment K by an equation of the form

$$P = cL^{\alpha}K^{\beta}$$

where c is a constant of proportionality and α and β are constants such that $0 < \alpha < 1$ and $0 < \beta < 1$. This is called the ***Cobb-Douglas Production Model.*** Typically, P, L and K are expressed in their terms of their equivalent monetary values.

EXAMPLE1:

A toy manufacturer estimates a production function to be $f(x, y) = 100x^{0.6}y^{0.4}$. Compare the production level when $x = 1000$ and $y = 500$ with the production level when $x = 2000$ and $y = 1000$.

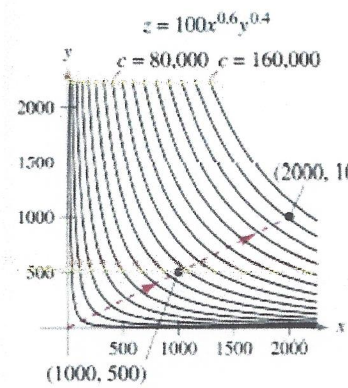
When $x = 1000$ and $y = 500$, the production level is

$$f(1000, 500) = 100\left(1000^{0.6}\right)(500)^{0.4} \approx 75,786.$$

When $x = 1000$ and $y = 500$, the production level is

$$f(2000, 1000) = 100\left(2000^{0.6}\right)(1000)^{0.4} \approx 151,572.$$

Note that by doubling both x and y, you double the production level.



Level curves (at increments of 10,000)

EXAMPLE 2:

The surfboard company you own has the Cobb-Douglas production function $P(x, y) = x^{0.3}y^{0.7}$ where P is the number of surfboards it produces per year, x is the number of employees, and the y is the daily operating budget (in dollars).

a) Find $\frac{dy}{dx}$

P is a constant

$$0 = \frac{d}{dx}\left(x^{0.3}y^{0.7}\right)$$

$$0 = 0.3x^{-0.7}y^{0.7} + x^{0.3}\left(0.7y^{-0.3}\right)\frac{dy}{dx}$$

$$\frac{-0.3y^{0.7}}{x^{0.7}} = \frac{0.7x^{0.3}}{y^{0.3}} \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{0.3y}{0.7x}$$

$$\boxed{\frac{dy}{dx} = -\frac{3y}{7x}}$$

y → daily budget in $

x → # of employees

b) Evaluate the derivative at $x = 30$ and $y = 10,000$ and interpret the answer.

$$\left.\frac{dy}{dx}\right|_{\substack{x=30\\ y=10,000}} = -\frac{3(10,000)}{7(30)} \approx -143 \Rightarrow -\$143 \text{ per employee}$$

• Daily budget is decreasing $143 per each additional employee at employment level of 30 employees and a daily operating budget of $10,000.

⇒ Increasing workforce by 1 worker will <u>save</u> approx. $143 per day

(X)

EXAMPLE 3:

The Jim Saki Company manufactures cotton athletic socks. Production is partially automated through the use of robots. Daily operating costs amount to \$50 per laborer and \$30 per robot. The number of pairs of socks the company can manufacture in a day is given by a Cobb-Douglas production formula

$$q = 50n^{0.6}r^{0.4}$$

where q is the number of pairs of socks that can be manufactured by n laborers and r robots. Assuming that the company wishes to produce 1.000 pairs of socks per day at a minimum cost, how many laborers and how many robots should it use?

The objective is to minimize the daily cost $C = 50n + 30r$ with constraints given by the daily quota of 1,000 and the fact that n and r are nonnegative.

$$1{,}000 = 50n^{0.6}r^{0.4} \Rightarrow n^{0.6} = \frac{1{,}000}{50r^{0.4}}$$

$$\left(n^{0.6}\right)^{1/0.6} = \left(\frac{1{,}000}{50r^{0.4}}\right)^{1/0.6} \quad \text{so} \quad n = \left(\frac{20}{r^{0.4}}\right)^{1/0.6} \Rightarrow \frac{20^{1/0.6}}{r^{4/0.6}} \Rightarrow \frac{20^{5/3}}{r^{2/3}} \approx \frac{147.36}{r^{2/3}}$$

Now: $C(r) \approx 50\left(\frac{147.36}{r^{2/3}}\right) + 30r = 7{,}368r^{-2/3} + 30r$

The remaining constraint is that $r > 0$

To find the minimum value of $C(r)$, take the derivative and set it equal to 0.

$$C'(r) \approx -4{,}912r^{-5/3} + 30 = 0 \text{ when } r \approx (0.006107)^{-3/5} \approx 21.3$$

Hence, the cost is minimized at about $C(21.3) \approx \$1{,}600$

Problems:

1. The number of CDs per hour that *Snappy Sounds* can manufacture at its plant is given by $P = x^{0.6}y^{0.4}$ where x is the number of workers at the plant and y is the monthly budget (in dollars). Assume P is constant, and compute $\frac{dy}{dx}$ and interpret the results when $x = 100$ and $y = 200{,}000$.

   Answer: – \$3,000 per worker. The monthly budget to maintain production at the fixed level P is decreasing by approximately \$3,000 per additional worker at an employment level of 100 workers and a monthly operating budget of \$200,000

2. Your automobile assembly plant has a Cobb-Douglas production function given by $q = x^{0.5}y^{0.5}$ where q is the number of automobiles it produces per year, x is the number of employees, and y is the daily operating budget (in dollars). Annual operating costs amount to an average of \$20,000 per employee plus the operating budget of \$365y. Assume you wish to produce 1,000 automobiles per year at a minimum cost. How many employees should you hire?

   Answer: Minimize cost $C = 20{,}000x + 365y$ subject to $x^{0.5}y^{0.5} = 1{,}000$. C has a minimum at $x \approx 135$, so you should hire 135 employees.

Cobb-Douglas Production Models

(Why do we care about implicit differentiation?)

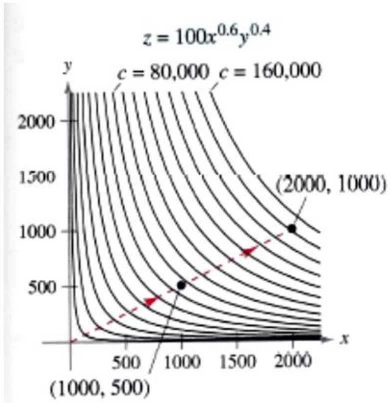
In economics, a production model is a mathematical relationship between the output of a company or a country and the labor and capital equipment required to produce that output. Much of the pioneering work in the field of production models occurred in the 1920s when Paul Douglas of the University of Chicago and his collaborator Charles Cobb proposed that the output *P* can be expressed in terms of the labor *L* and the capital equipment *K* by an equation of the form

$$P = cL^{\alpha}K^{\beta}$$

where *c* is a constant of proportionality and α and β are constants such that $0 < \alpha < 1$ and $0 < \beta < 1$. This is called the ***Cobb-Douglas Production Model.*** Typically, *P, L* and *K* are expressed in their terms of their equivalent monetary values.

EXAMPLE1:

A toy manufacturer estimates a production function to be $f(x,y) = 100x^{0.6}y^{0.4}$. Compare the production level when $x = 1000$ and $y = 500$ with the production level when $x = 2000$ and $y = 1000$.



Level curves (at increments of 10,000)

When $x = 1000$ and $y = 500$, the production level is

$$f(1000,\ 500) = 100\left(1000^{0.6}\right)(500)^{0.4} \approx 75{,}786.$$

When $x = 1000$ and $y = 500$, the production level is

$$f(2000,\ 1000) = 100\left(2000^{0.6}\right)(1000)^{0.4} \approx 151{,}572.$$

Note that by doubling both *x* and *y*, you double the production level.

EXAMPLE 2:

The surfboard company you own has the Cobb-Douglas production function $P(x,y) = x^{0,3}y^{0.7}$ where *P* is the number of surfboards it produces per year, *x* is the number of employees, and the *y* is the daily operating budget (in dollars).

a) Find $\dfrac{dy}{dx}$

b) Evaluate the derivative at $x = 30$ and $y = 10{,}000$ and interpret the answer.

EXAMPLE 3:

The Jim Saki Company manufactures cotton athletic socks. Production is partially automated through the use of robots. Daily operating costs amount to \$50 per laborer and \$30 per robot. The number of pairs of socks the company can manufacture in a day is given by a Cobb-Douglas production formula

$$q = 50n^{0.6}r^{0.4}$$

where q is the number of pairs of socks that can be manufactured by n laborers and r robots. Assuming that the company wishes to produce 1.000 pairs of socks per day at a minimum cost, how many laborers and how many robots should it use?

The objective is to minimize the daily cost $C = 50n + 30r$ with constraints given by the daily quota of 1,000 and the fact that n and r are nonnegative.

$$1{,}000 = 50n^{0.6}r^{0.4} \quad \Rightarrow \quad n^{0.6} = \frac{1{,}000}{50r^{0.4}}$$

$$\left(n^{0.6}\right)^{1/0.6} = \left(\frac{1{,}000}{50r^{0.4}}\right)^{1/0.6} \quad \text{so} \quad n = \left(\frac{20}{r^{0.4}}\right)^{1/0.6} \Rightarrow \frac{20^{1/0.6}}{r^{4/0.6}} \Rightarrow \frac{20^{5/3}}{r^{2/3}} \approx \frac{147.36}{r^{2/3}}$$

Now: $C(r) \approx 50\left(\dfrac{147.36}{r^{2/3}}\right) + 30r = 7{,}368r^{-2/3} + 30r$

The remaining constraint is that $r > 0$

To find the minimum value of $C(r)$, take the derivative and set it equal to 0.

$$C'(r) \approx -4{,}912r^{-5/3} + 30 = 0 \quad \text{when} \quad r \approx (0.006107)^{-3/5} \approx 21.3$$

Hence, the cost is minimized at about $C(21.3) \approx \$1{,}600$

Problems:

1. The number of CDs per hour that *Snappy Sounds* can manufacture at its plant is given by $P = x^{0.6}y^{0.4}$ where x is the number of workers at the plant and y is the monthly budget (in dollars). Assume P is constant, and compute $\dfrac{dy}{dx}$ and interpret the results when $x = 100$ and $y = 200{,}000$.

   Answer: – \$3,000 per worker. The monthly budget to maintain production at the fixed level P is decreasing by approximately \$3,000 per additional worker at an employment level of 100 workers and a monthly operating budget of \$200,000

2. Your automobile assembly plant has a Cobb-Douglas production function given by $q = x^{0.5}y^{0.5}$ where q is the number of automobiles it produces per year, x is the number of employees, and y is the daily operating budget (in dollars). Annual operating costs amount to an average of \$20,000 per employee plus the operating budget of \$365y. Assume you wish to produce 1,000 automobiles per year at a minimum cost. How many employees should you hire?

   Answer: Minimize cost $C = 20{,}000x + 365y$ *subject to* $x^{0.5}y^{0.5} = 1{,}000$. C *has a minimum at* $x \approx 135$*, so you should hire 135 employees.*

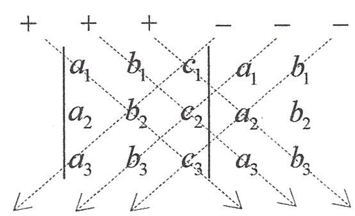
Applications of the Determinant

Determinants can be used to solve systems of linear equations as well as determining if a matrix has an inverse. They are also useful in many other situations, for example, in the computation of the cross product of vectors. The cross product of two vectors produces a vector which is orthogonal to the two vectors. If the cross product is 0, it confirms parallel vectors. Its magnitude produces the area of the parallelogram have the two vectors as adjacent sides. The magnitude of the cross product of two vectors divided by the product of their magnitudes produces the sine of the angle between the two vectors. The theory of determinants is rather attractive and deserves study on its own merits.

If a matrix is an *nth*-order matrix (*n* rows and *n* columns) its determinant is an *nth*-order determinant.

- The value of a second-order determinant is given by: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2$.
- For a 3^{rd}-order matrix, the process is similar, but more detailed:

Method 1: $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3$$



$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

- Another way to compute this determinant is by minors and cofactors. Block out the row and column of cofactor a_1 and multiply a_1 by its 2nd second order determinant. Do the same with the cofactors b_1 and c_1. The cofactor is also multiplied by $(-1)^{i+j}$ where *i* is the row of the cofactor, and *j* is the column of it.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \qquad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

EXAMPLE 1: Find the value of the determinant:

Method 1: $$\begin{vmatrix} 3 & 2 & 7 \\ -1 & 5 & 3 \\ 2 & -3 & -6 \end{vmatrix} = 3\cdot5\cdot(-6) + 2\cdot3\cdot2 + 7\cdot(-1)(-3) - 2\cdot5\cdot7 - (-3)\cdot3\cdot3 - (-6)(-1)\cdot2$$

$$= -90 + 12 + 21 - 70 + 27 - 12 = -112$$

Method 2: $$\begin{vmatrix} 3 & 2 & 7 \\ -1 & 5 & 3 \\ 2 & -3 & -6 \end{vmatrix} = 3\cdot\begin{vmatrix} 5 & 3 \\ -3 & -6 \end{vmatrix} - 2\cdot\begin{vmatrix} -1 & 3 \\ 2 & -6 \end{vmatrix} + 7\cdot\begin{vmatrix} -1 & 3 \\ 2 & -3 \end{vmatrix}$$

$$= 3\cdot(-30+9) - 2\cdot(6-6) + 7\cdot(3-10) = 3(-21) + 0 + 7(-7) = -63-49 = -112$$

Cramer's Rule is rarely taught in Algebra 2 and Pre-Calculus courses now, but using determinants and the process of expanding my minors to solve systems of equations is easy, and it will spark a procedural memory in later when these students work with the cross product, partial derivatives, gradients, del operators and Jacobians.

EXAMPLE 2: Solve the system using Cramer's Rule

$$\begin{cases} 2x - 3y + 4z = 1 \\ x \qquad\quad + 6z = 0 \\ 3x - 2y \qquad\ \ = 5 \end{cases}$$

$|D|$ is the determinant of the coefficient matrix.

$|D_x|$ is the determinant of the matrix with the constant column replacing the coefficients of x.

$|D_y|$ is the determinant of the matrix with the constant column replacing the coefficients of y.

$|D_z|$ is the determinant of the matrix with the constant column replacing the coefficients of z.

The variables are found by these ratios: $x = \frac{|D_x|}{|D|}$ $\quad y = \frac{|D_y|}{|D|}$ $\quad z = \frac{|D_z|}{|D|}$

$$|D| = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 0 & 6 \\ 3 & -2 & 0 \end{vmatrix} = -38 \qquad |D_x| = \begin{vmatrix} 1 & -3 & 4 \\ 0 & 0 & 6 \\ 5 & -2 & 0 \end{vmatrix} = -78$$

$$|D_y| = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 0 & 6 \\ 3 & 5 & 0 \end{vmatrix} = -22 \qquad |D_z| = \begin{vmatrix} 2 & -3 & 1 \\ 1 & 0 & 0 \\ 3 & -2 & 5 \end{vmatrix} = 13$$

$$x = \frac{|D_x|}{|D|} = \frac{-78}{-38} = \frac{39}{19} \qquad y = \frac{|D_y|}{|D|} = \frac{-22}{-38} = \frac{11}{19} \qquad z = \frac{|D_z|}{|D|} = \frac{13}{-38} = -\frac{13}{38}$$

Considering the messy solutions to this system of equations, this is by far an easier way to solve the system rather than other methods such as substitution, Gaussian elimination, or inverses and matrix equations. And teaching this simple method will make the process of working with vectors easier as students continue in their mathematical skills and analysis.

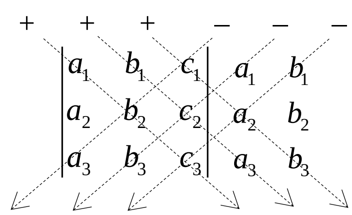
Applications of the Determinant

Determinants can be used to solve systems of linear equations as well as determining if a matrix has an inverse. They are also useful in many other situations, for example, in the computation of the cross product of vectors. The cross product of two vectors produces a vector which is orthogonal to the two vectors. If the cross product is 0, it confirms parallel vectors. Its magnitude produces the area of the parallelogram have the two vectors as adjacent sides. The magnitude of the cross product of two vectors divided by the product of their magnitudes produces the sine of the angle between the two vectors. The theory of determinants is rather attractive and deserves study on its own merits.

If a matrix is an *nth*-order matrix (*n* rows and *n* columns) its determinant is an *nth*-order determinant.

- The value of a second-order determinant is given by: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2$.
- For a *3rd*-order matrix, the process is similar, but more detailed:

Method 1: $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3$$



$$= a_1\left(b_2c_3 - b_3c_2\right) - b_1\left(a_2c_3 - a_3c_2\right) + c_1\left(a_2b_3 - a_3b_2\right)$$

- Another way to compute this determinant is by minors and cofactors. Block out the row and column of cofactor a_1 and multiply a_1 by its 2nd second order determinant. Do the same with the cofactors b_1 and c_1. The cofactor is also multiplied by $(-1)^{i+j}$ where *i* is the row of the cofactor, and *j* is the column of it.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \qquad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$= a_1\left(b_2c_3 - b_3c_2\right) - b_1\left(a_2c_3 - a_3c_2\right) + c_1\left(a_2b_3 - a_3b_2\right)$$

EXAMPLE 1: Find the value of the determinant:

Method 1: $$\begin{vmatrix} 3 & 2 & 7 \\ -1 & 5 & 3 \\ 2 & -3 & -6 \end{vmatrix} = 3\cdot 5\cdot(-6) + 2\cdot 3\cdot 2 + 7\cdot(-1)(-3) - 2\cdot 5\cdot 7 - \left(-3\right)\cdot 3\cdot 3 - (-6)(-1)\cdot 2$$

$$= -90 + 12 + 21 - 70 + 27 - 12 = -112$$

Method 2: $$\begin{vmatrix} 3 & 2 & 7 \\ -1 & 5 & 3 \\ 2 & -3 & -6 \end{vmatrix} = 3\cdot\begin{vmatrix} 5 & 3 \\ -3 & -6 \end{vmatrix} - 2\cdot\begin{vmatrix} -1 & 3 \\ 2 & -6 \end{vmatrix} + 7\cdot\begin{vmatrix} -1 & 3 \\ 2 & -3 \end{vmatrix}$$

$$= 3\cdot(-30+9) - 2\cdot(6-6) + 7\cdot(3-10) = 3(-21) + 0 + 7(-7) = -63-49 = -112$$

Cramer's Rule is rarely taught in Algebra 2 and Pre-Calculus courses now, but using determinants and the process of expanding my minors to solve systems of equations is easy, and it will spark a procedural memory in later when these students work with the cross product, partial derivatives, gradients, del operators and Jacobians.

EXAMPLE 2: Solve the system using Cramer's Rule

$$\begin{cases} 2x - 3y + 4z = 1 \\ x \qquad\quad + 6z = 0 \\ 3x - 2y \qquad\ = 5 \end{cases}$$

$|D|$ is the determinant of the coefficient matrix.

$|D_x|$ is the determinant of the matrix with the constant column replacing the coefficients of *x*.

$|D_y|$ is the determinant of the matrix with the constant column replacing the coefficients of *y*.

$|D_z|$ is the determinant of the matrix with the constant column replacing the coefficients of *z*.

The variables are found by these ratios: $x = \frac{|D_x|}{|D|} \qquad y = \frac{|D_y|}{|D|} \qquad z = \frac{|D_z|}{|D|}$

$$|D| = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 0 & 6 \\ 3 & -2 & 0 \end{vmatrix} = -38 \qquad |D_x| = \begin{vmatrix} 1 & -3 & 4 \\ 0 & 0 & 6 \\ 5 & -2 & 0 \end{vmatrix} = -78$$

$$|D_y| = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 0 & 6 \\ 3 & 5 & 0 \end{vmatrix} = -22 \qquad |D_z| = \begin{vmatrix} 2 & -3 & 1 \\ 1 & 0 & 0 \\ 3 & -2 & 5 \end{vmatrix} = 13$$

$$x = \frac{|D_x|}{|D|} = \frac{-78}{-38} = \frac{39}{19} \qquad y = \frac{|D_y|}{|D|} = \frac{-22}{-38} = \frac{11}{19} \qquad z = \frac{|D_z|}{|D|} = \frac{13}{-38} = -\frac{13}{38}$$

Considering the messy solutions to this system of equations, this is by far an easier way to solve the system rather than other methods such as substitution, Gaussian elimination, or inverses and matrix equations. And teaching this simple method will make the process of working with vectors easier as students continue in their mathematical skills and analysis.

NAME___________________________

PROJECT: VOLUMES BY SLICING
(Created by Wanda Savage)

We have just completed the numerical computation of the volume of solids with a known cross-section. Your assignment is to make a model of one. You may consult the examples in your notes and on p. 418-419 for further clarification of this assignment.

You have been assigned **PROBLEM #______________**. ***This sheet must accompany your project.***

Point distribution will be allotted as follows. This project will be counted as a test grade. It will be due (Date) . The solid must be on a base which is <u>no larger than 6"x6"</u>. Your solid must have <u>at least 20 cross-sections</u>. You must also completely and correctly work out the numerical volume of your solid *(That is, you must set up the integral correctly and work it out thoroughly)*.

		PTS	POINTS RECEIVED
ORIGINALITY:		***15 pts.***	
	MATERIALS	5	__________
	PRESENTATION	10	__________
APPEARANCE:		***50 pts.***	
	BASE of Solid:	**10 pts.**	
	Accurate Shape of Base 5		__________
	Correct scale marked	5	__________
	CROSS-SECTIONS:	**40 PTS**	
	Secure	5	__________
	Accurate Shape 15		__________
	Visual Color	5	__________
	Completeness of the shape of the solid	15	__________
USEFULNESS AS A MODEL:		***15 pts.***	__________
EXTENSION:		***20 pts.***	
	Overall Construction	10	__________
	Correct working of the problem on paper	10	__________

SOLID VARIETIES

(Created by Wanda Savage)

Cross-Sections are $\perp$ to the BASES BOUNDED BY:

Type I:

1. $y = x+1$ and $y = x^2 - 1$, cross-sections are squares, $\perp$ to $x-axis$.
2. $y = x+1$ and $y = x^2 - 1$, cross-sections are equilateral triangles, $\perp$ to $x-axis$. (See front cover of your book for this formula.)
3. $y = x+1$ and $y = x^2 - 1$, cross-sections are rectangles of height 1, $\perp$ to $x-axis$.
4. $y = x+1$ and $y = x^2 - 1$, cross-sections are semi-ellipses of height 2, $\perp$ to $x-axis$. (See front cover of your book for this formula.)

Type II:

5. $y = x^3$, $y = 0$ and $x = 1$, cross-sections are equilateral triangles, $\perp$ to $y-axis$.
6. $y = x^3$, $y = 0$ and $x = 1$, cross-sections are squares, $\perp$ to $x-axis$.
7. $y = x^3$, $y = 0$, and $x = 1$, cross-sections are trapezoids for which $h = b_1 = \frac{1}{2}b_2$ where b_1 and b_2 are upper and lower bases, $\perp$ to $y-axis$.
8. $y = x^3$, $y = 0$, and $x = 1$, cross-sections are semi-circles, $\perp$ to $y-axis$.
9. $y = x^3$, $y = 0$, and $x = 1$, cross-sections are semi-ellipses whose heights are twice the lengths of their bases, $\perp$ to $y-axis$. (See front cover of your book for this formula.)

Type III:

10. $x = y^2$ and $x = 9$, cross-sections are squares, $\perp$ to $x-axis$.
11. $x = y^2$ and $x = 9$, cross-sections are quarter-circles, $\perp$ to $x-axis$.
12. $x = y^2$ and $x = 9$, cross-sections are rectangles of height 2, $\perp$ to $x-axis$.
13. $x = y^2$ and $x = 9$, cross-sections are equilateral triangles, $\perp$ to $x-axis$.
14. $x = y^2$ and $x = 9$, cross-sections are triangles with $h = \frac{1}{4}b$, $\perp$ to $x-axis$.
15. $x = y^2$ and $x = 9$, cross-sections are trapezoids with lower base in $xy-$plane, upper base $= \frac{1}{2}$ lower base, $h = \frac{1}{4}$ lower base, $\perp$ to $x-axis$.
16. $x = y^2$ and $x = 9$, cross-sections are semi-circles, $\perp$ to $x-axis$.

Type IV:

17. circle, $x^2 + y^2 = 4$, cross-sections are isosceles triangles with $h = b$, (triangle base is in the $xy-$plane), $\perp$ to $x-axis$.
18. circle, $x^2 + y^2 = 4$, cross-sections are semi-circles, $\perp$ to $x-axis$.
19. circle, $x^2 + y^2 = 4$, cross-sections are squares, $\perp$ to $x-axis$.
20. circle, $x^2 + y^2 = 4$, cross-sections are equilateral triangles, $\perp$ to $x-axis$.
21. circle, $x^2 + y^2 = 4$, cross-sections are isosceles right triangles, (right angle formed at the $xy-$plane), $\perp$ to $x-axis$.

<u>Type V</u>:

22. $x = y^2$ and $x = 3 - 2y^2$, cross-sections are rectangles of height 2, $\perp$ to $x - axis$.
23. $x = y^2$ and $x = 3 - 2y^2$, cross-sections are equilateral triangles, $\perp$ to $x - axis$.
24. $x = y^2$ and $x = 3 - 2y^2$, cross-sections are isosceles right triangles, (hypotenuse in $xy -$ plane), $\perp$ to $x - axis$.

<u>Type VI</u>:

25. $y = x$ and $y^2 = x$, cross-sections are semi-circles, $\perp$ to $x - axis$.
26. $y = x$ and $y = x^2$, cross-sections are semi-circles, $\perp$ to $x - axis$.

<u>Type VII</u>:

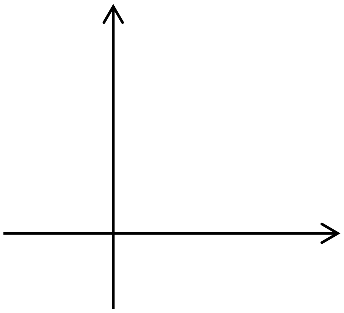
27. $y = 4$ and $y = x^2$, cross-sections are squares, $\perp$ to $x - axis$.
28. $y = 4$ and $y = x^2$, cross-sections are isosceles right triangles, (right angle at $y = 4$), $\perp$ to $x - axis$.
29. $y = 4$ and $y = x^2$, cross-sections are isosceles right triangles, (right angle at $y = x^2$), $\perp$ to $x - axis$.

<u>Type VIII</u>: (Varied functions)

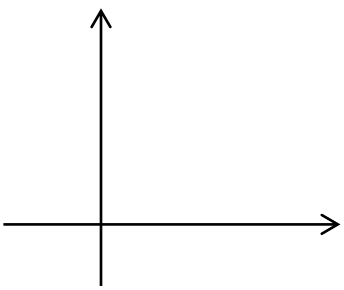
30. one arch of $y = \cos x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and the $x - axis$, cross-sections are squares, $\perp$ to $x - axis$.
31. $y^2 = 4x$ and $x = 4$, cross-sections are semi-circles, $\perp$ to $y - axis$.
32. $y = 1 - x^2$ and $y = 1 - x^4$, cross-sections are squares, $\perp$ to $x - axis$.
33. $x^2 = 18y$ and $y = 2$, cross-sections are squares, $\perp$ to $y - axis$.

How Sweet It Is!!!
Finding volumes of solids by revolution
(Created by Dixie Ross)

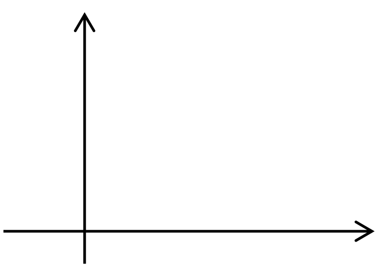
1. Rotate the region enclosed by $y=\sqrt{\sin x}$ and $y=0$ on the interval $[0, \pi]$ about the *x*-axis. Identify the shape of the solid formed and determine its volume.



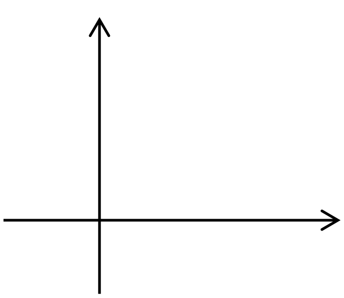
2. Rotate the region enclosed by $y=x^2$, $y=0$ and $x=2$ about the *x*-axis. Identify the shape of the solid formed and determine its volume.



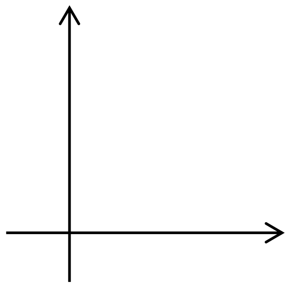
3. Consider the region enclosed by $y=\frac{1}{2}x-1$, $x=0$, $y=0$ and $y=2$. Identify the shape of the solid formed when this region is revolved about the *x*-axis and determine its volume.



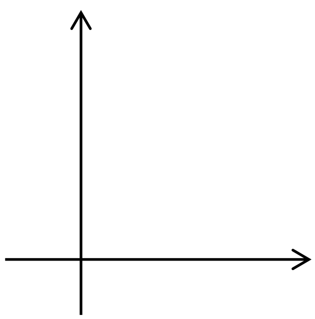
4. Consider the region in the <u>first quadrant</u> enclosed by $y=4-x^2$. Identify the shape of the solid formed when this region is revolved about the *x*-axis and determine its volume.



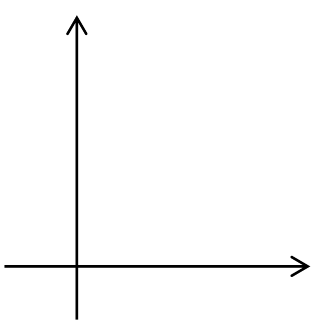
5. Consider the region in the first quadrant enclosed by $y = 4 - x^2$. Identify the shape of the solid formed when this region is revolved about the *y*-axis and determine its volume.



6. Rotate the region enclosed by $y = 4 - x^2$ and the *x*-axis about the *x*-axis. Identify the shape of the solid formed when this region is revolved about the *x*-axis and determine its volume.



7. Consider the region enclosed by $y = x^2$, $y = 0$ and $x = 3$. Identify the solid formed when this region is revolved about the line $x = 3$ and determine its volume.



Circuit – Writing and Interpreting Riemann Sums Name_______________________

Directions: Beginning in the first cell marked #1, find the requested information. To advance in the circuit, hunt for your answer and mark that cell #2. Continue working in this manner until you complete the circuit. For all the following problems use a right hand Riemann sum with equal partitions.

__1__ Ans: $\int_3^5 \sqrt{x^3+1}\,dx$ Find a limit equal to $\int_0^4 (x^2+1)dx$.	______ Ans: $\lim_{n\to\infty}\sum_{k=1}^{n} \ln\left(\frac{8}{n}k+1\right)\frac{4}{n}$ Find a limit equal to $\int_3^7 (x^2-8)dx$.
______ Ans: $\int_2^4 (x^2+1)dx$ Find an integral expression equal to: $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(\frac{2}{n}k\right)^3+3\right)\frac{2}{n}$.	______ Ans: $\int_0^2 (x^3+2)dx$ Find an integral expression equal to: $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sqrt{\left(\frac{2}{n}k+3\right)+1}\right)\frac{2}{n}$.

______ Ans: $\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(\frac{4}{n}k+1\right)\left(\frac{4}{n}\right)$

Find an integral expression equal to:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(\frac{2}{n}k+3\right)+1\right)\frac{2}{n}.$$

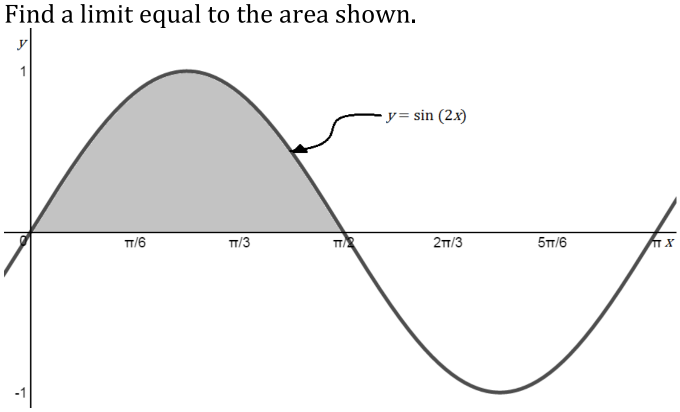
______ Ans: $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sqrt{\frac{16}{n}k+9}\right)\frac{4}{n}$

Find an integral expression equal to:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(\frac{2}{n}k\right)^2+3\right)\frac{2}{n}.$$

______ Ans: $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{16}{n^2}k^2+\frac{24}{n}k+1\right)\frac{4}{n}$

Find a limit equal to the area shown.



______ Ans: $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{16}{n^2}k^2+1\right)\frac{4}{n}$

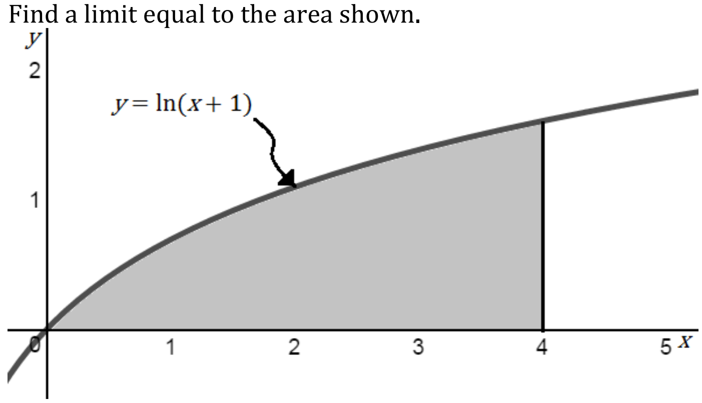
Find a limit equal to $\int_0^{\pi/2}\cos(2x)\,dx$.

______ Ans: $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\cos\left(\frac{\pi}{n}k\right)\right)\frac{\pi}{2n}$

Find a limit equal to $\int_3^7 \sqrt{4x-3}\,dx$.

______ Ans: $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sqrt{\frac{4}{n}k+2}\right)\frac{4}{n}$

Find a limit equal to the area shown.



______ Ans: $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sin\left(\frac{\pi}{n}k\right)\right)\frac{\pi}{2n}$

Find an integral expression equal to:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(\frac{2}{n}k\right)^3+2\right)\frac{2}{n}.$$

______ Ans: $\int_3^5 (x+1)dx$

Find an integral expression equal to:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sqrt{\left(\frac{2}{n}k+3\right)^3+1}\right)\frac{2}{n}.$$

______ Ans: $\int_3^5 \sqrt{x+1}\,dx$ Find an integral expression equal to: $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(\frac{2}{n}k+2\right)^2+1\right)\frac{2}{n}.$	______ Ans: $\int_0^2 (x^2+3)dx$ Find an integral expression equal to: $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sqrt{\left(\frac{2}{n}k+3\right)^2+1}\right)\frac{2}{n}.$
______ Ans: $\int_3^5 \sqrt{x^2+1}\,dx$ Find a limit equal to $\int_1^5 \ln(2x-1)\,dx$.	______ Ans: $\int_0^2 (x^3+3)\,dx$ Find a limit equal to the area shown.