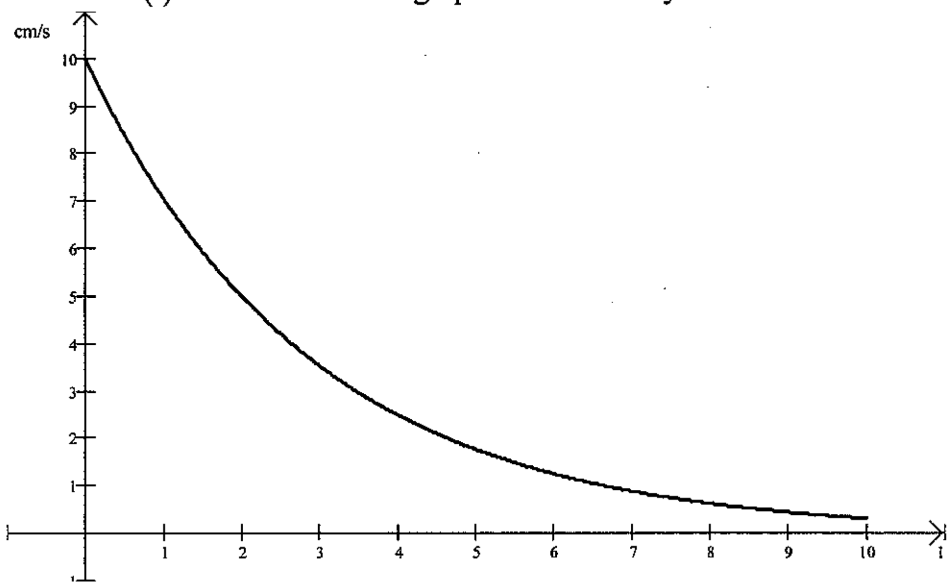
Integrating Velocity

Suppose that the horizontal velocity of an object on the time interval $0 \le t \le 10$ s is determined to be $v(t) = 10 \cdot 2^{-t/2}$ ft/s. A graph of the velocity is shown below.



Our goal is to determine the distance traveled by the object on the interval [0,10].

a. Explain how I know that the object could have traveled no more than 100 ft and no less than 3.125 ft.

b. Using two equal time intervals, determine an upper and lower bound for the distance that the car traveled.

c. Using five equal time intervals, determine an upper and lower bound for the distance that the car traveled.

d. Using ten equal time intervals, determine an upper and lower bound for the distance that the car traveled.

e. Develop a summation for the upper bound distance from n time intervals.

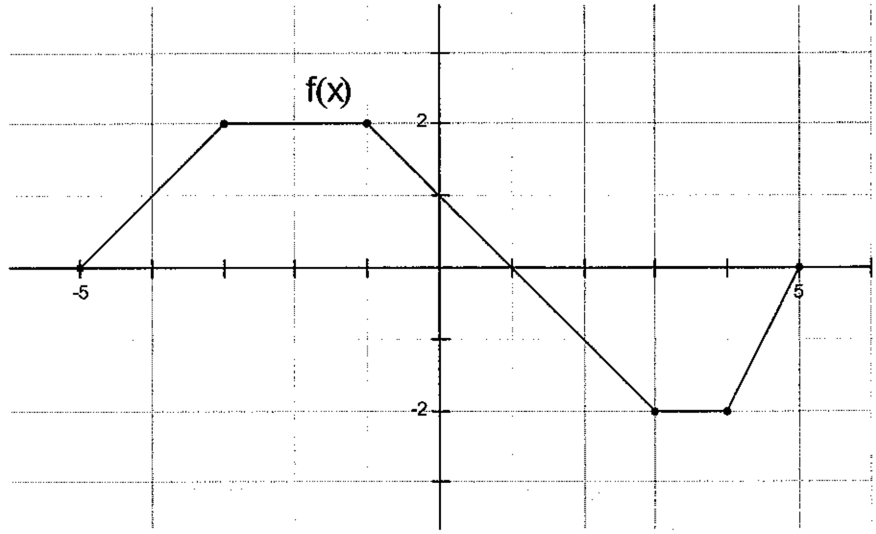
f. Develop a summation for the lower bound distance from n time intervals.

g. Take the limit of both upper and lower bound estimates as n approaches infinity.

h. What have you learned?

Basic Integration

1. Consider the function $f(x)$ shown in the graph below. Use the graph to find the integrals that follow.



a. $\int_{-3}^{-3} f(x)\,dx$

b. $\int_{-5}^{1} f(x)\,dx$

c. $\int_{-5}^{3} f(x)\,dx$

d. $\int_{-5}^{5} f(x)\,dx$

e. $\int_{-3}^{-5} f(x)\,dx$

f. $\int_{5}^{1} f(x)\,dx$

g. $\int_{5}^{-5} f(x)\,dx$

h. $\int_{-5}^{5} |f(x)|\,dx$

i. $\int_{-5}^{1} (f(x)+2)\,dx$

j. $\int_{-5}^{1} 3f(x)\,dx$

k. $\int_{-5}^{5} (3+|f(x)|)\,dx$

Generalizations Now make some generalizations about integration.

2. Suppose that the following is known about a function *f*:

$$\int_0^3 f(x)dx = 4 \qquad \text{and} \qquad \int_3^6 f(x)dx = -1$$

Find the following integrals.

a. $\int_0^6 f(x)dx$

b. $\int_6^3 f(x)dx$

c. $\int_0^3 4f(x)dx$

d. $\int_3^3 f(x)dx$

3. Evaluate the following integrals by making a graph of the function over the relevant interval.

a. $\int_{-3}^5 3\,dx$

b. $\int_1^4 x\,dx$

c. $\int_{-2}^4 (2-x)dx$

d. $\int_0^6 |x-3|\,dx$

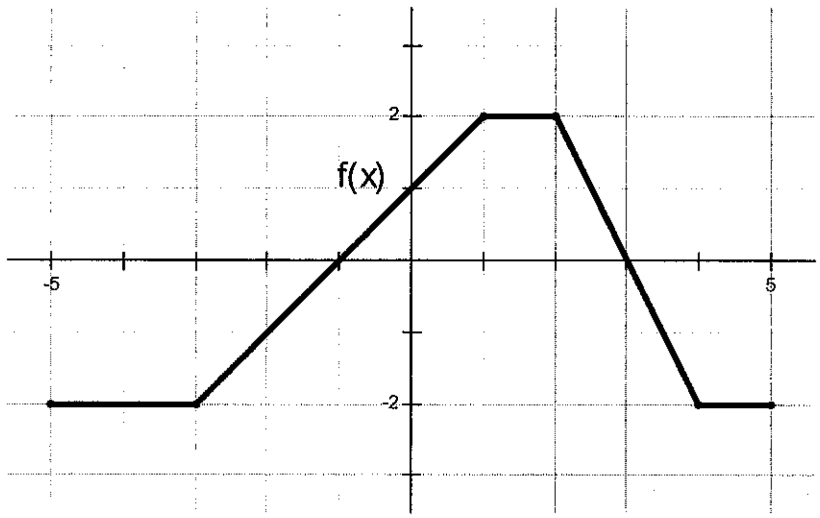
e. $\int_{-4}^4 \sqrt{16-x^2}\,dx$

f. $\int_0^3 \left(3-\sqrt{9-x^2}\right)dx$

The Area Function

Now that we've learned what an integral is, let's develop a new type of function: the area function.

Consider the function $f(x)$ defined on the interval [-5,5] shown below:



Now, a new function $A(t)$ is defined as the area under f from -3 to t:

$$A(t) = \int_{-3}^{t} f(x)dx$$

Find the following:

$A(-3) =$

$A(-1) =$

$A(1) =$

$A(3) =$

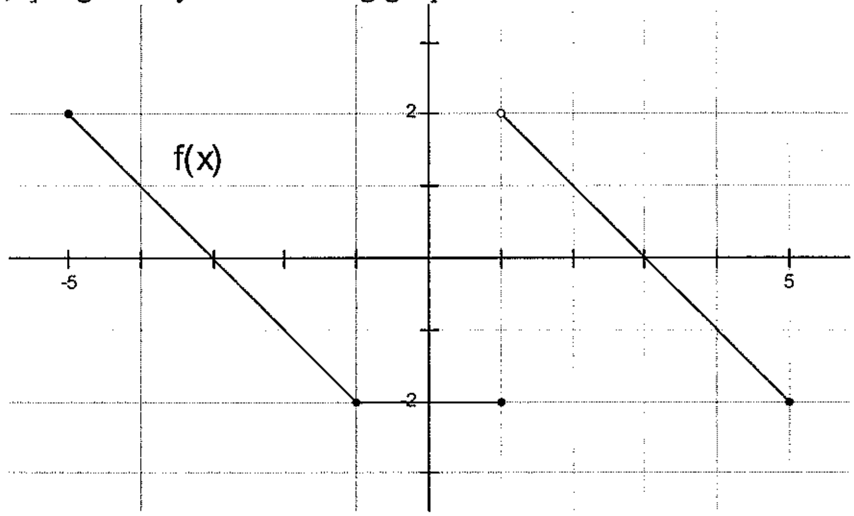
$A(5) =$

$A(-4) =$

$A(-5) =$

Sketch a graph of $A(t)$ on the grid above using the x-axis as the t-axis.

Now let's look at a different function. Suppose that a function $f(x)$ defined on the interval [-5,5] is given by the following graph:



Suppose that we define an area function as: $A(x) = \int_{-5}^{x} f(t)dt$

Why did I write $f(t)$ instead of $f(x)$?

Determine the following values of A.

$A(-5)$ $A(-4)$ $A(-3)$

$A(1)$ $A(3)$ $A(5)$

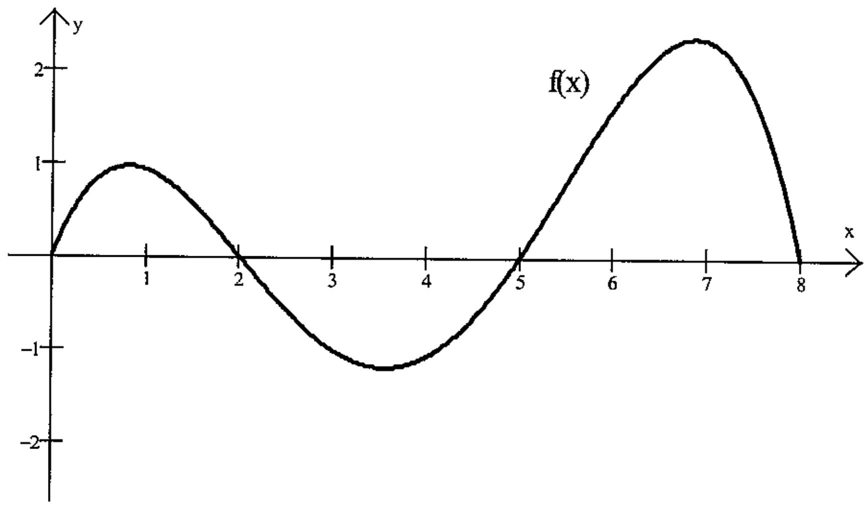
Sketch the graph of A on the grid above.

On what intervals is A increasing?

On what intervals is A decreasing?

At what values of x does A have local maximums or minimums?

For a new problem, consider the curve $f(x)$ defined on the interval [0,8] shown below.



Let g be defined by the integral $g(x) = \int_0^x f(t)\,dt$

What is $g(0)$?

Is $g(5) > 0$ or is $g(5) < 0$? Explain.

On what interval(s) is g increasing? On what interval(s) is g decreasing?

Make a rough sketch of g over the graph of f above.

At what value(s) of x does it appear that $g'(x) = 0$?

At what value of x on (0,8) does g have a local minimum? A local maximum?

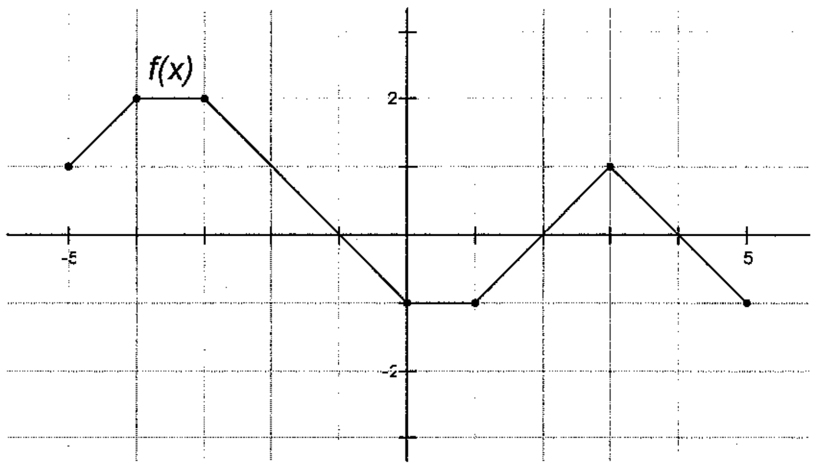
Does there appear to be a familiar relationship between g and f? What is this relationship?

Area Functions, a comparison

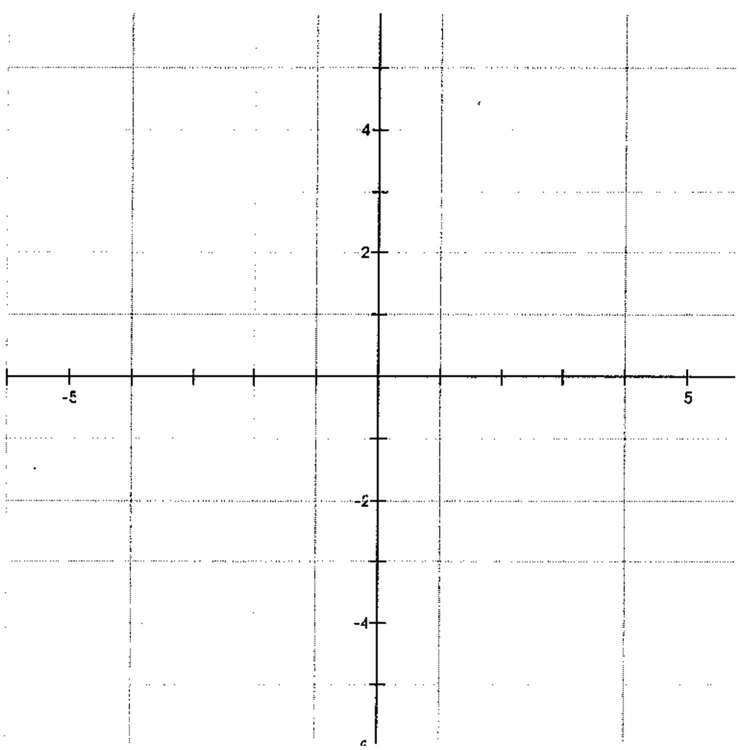
Question 1

Let's use a function *f* to create an area function: $A(x) = \int_{-5}^{x} f(t)\,dt$. Use geometry to complete the table below by finding values of *A* for integer values of *x* on the interval $-5 \le x \le 5$.

x	$A(x)$
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	



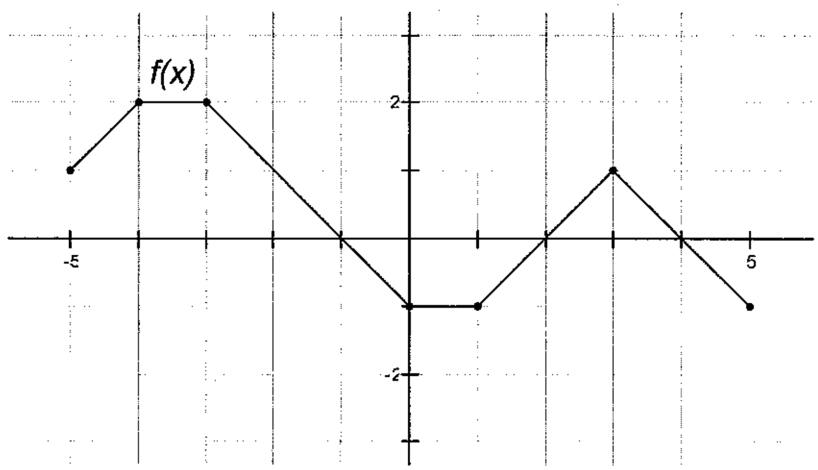
Now, on the grid below, sketch a graph of $A(x)$ on the interval $-5 \le x \le 5$.



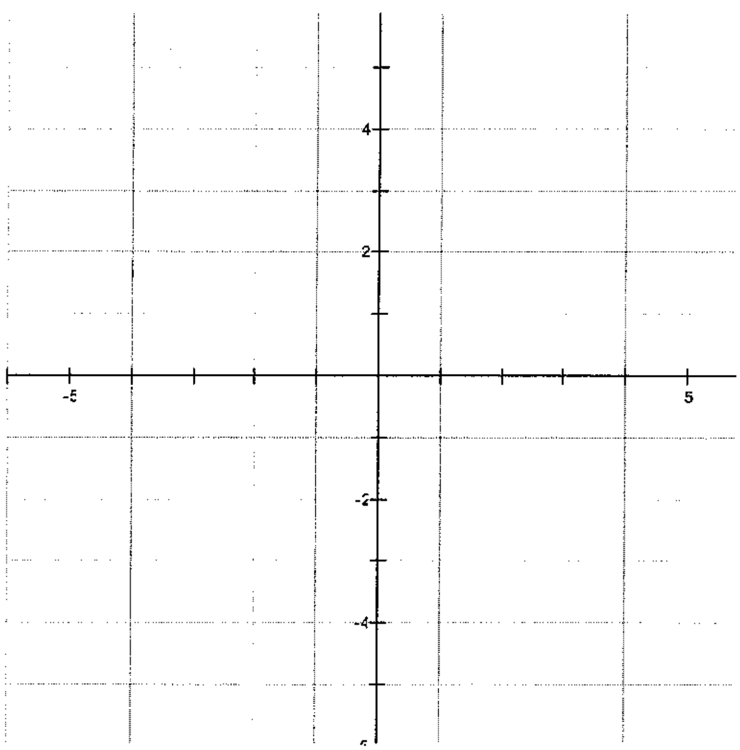
Question 2

Now, let's define a new area function: $B(x) = \int_{-3}^{x} f(t)\,dt$, where f is the same function as in question 1. Complete the table below by finding values of B for integer values of x on the interval $-5 \le x \le 5$.

x	$B(x)$
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	



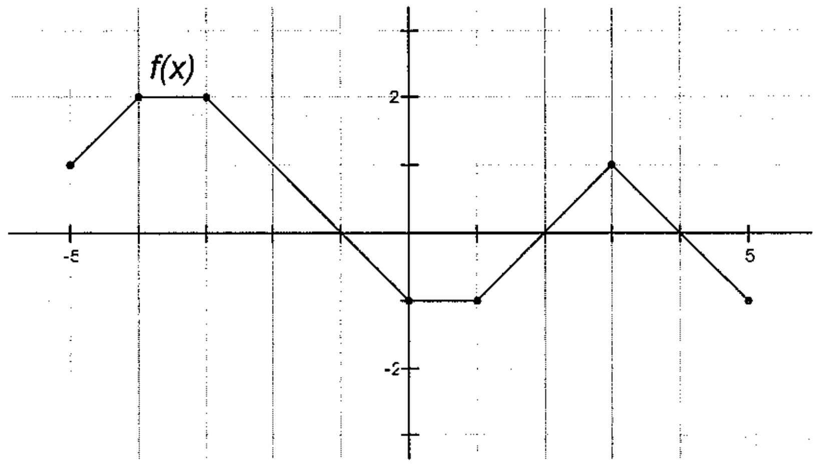
Now, on the grid below, sketch a graph of $B(x)$ on the interval $-5 \le x \le 5$.



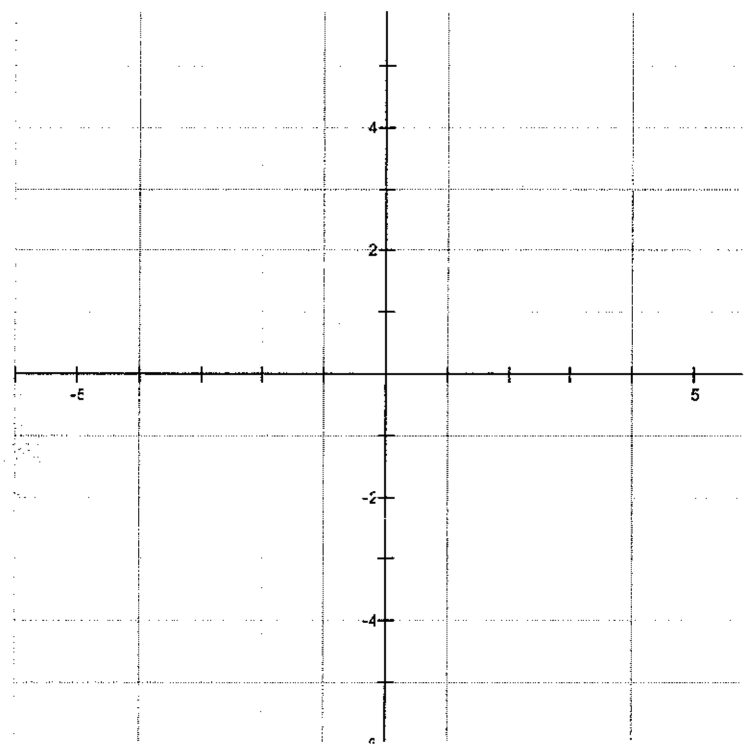
Question 3

Now, let's define a new area function: $C(x) = \int_1^x f(t)\,dt$, where f is the same function as in question 1. Complete the table below by finding values of C for integer values of x on the interval $-5 \le x \le 5$.

x	$C(x)$
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	



Now, on the grid below, sketch a graph of $C(x)$ on the interval $-5 \le x \le 5$.



4. Look at your graphs for the two area functions, *A*, *B* and *C*. How are the graphs similar?

5. Look at your graphs for the two area functions, *A*, *B* and *C*. How are the graphs different?

6. How is the graph of *f* related to the graph of *A*?

7. How is the graph of *f* related to the graph of *B*?

8. How is the graph of *f* related to the graph of *C*?

Using the Fundamental Theorem of Calculus

Write the Fundamental Theorem of Calculus in three ways.

Example 1 Consider the integral function $g(x)=\int_0^x 4\cos(t)e^{-t/5}\,dt$.

a. On the interval [0,10], on what intervals is g increasing? Explain.

b. At what values of x on (0,10) does g have a local minimum? Explain.

c. What is $g'(\pi)$? Explain.

d. What is $g''(\pi)$? Explain.

e. If $h(x)=\int_0^{3x} 4\cos(t)e^{-t/3}\,dt$, find $h'(\pi)$.

Example 2 Suppose that $f'(x)=\sqrt{1+\sin x}$ and $f(1)=-3$.

a. Find $f(6)$.

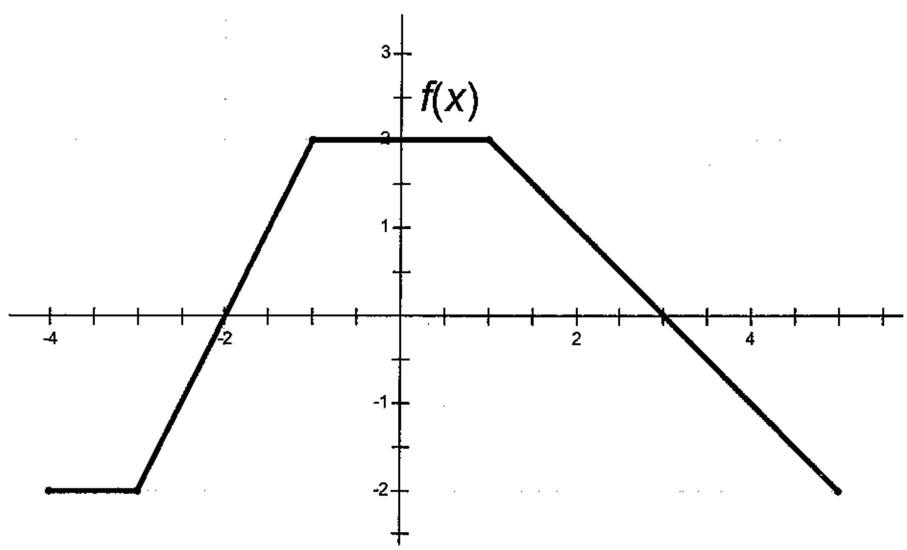
b. Find $f(-3)$.

Example 3 A function *f*(*x*) defined on the interval [-4,5] is shown in the graph below. Another function $g(x)$ is defined by the integral $g(x)=\int_{-1}^{x} f(t)dt$.

a. What is *g*(2)?

b. What is $g'(2)$?

c. What is $g''(2)$?



d. What is the maximum value of *g* on the interval [-4,5]? Justify your answer.