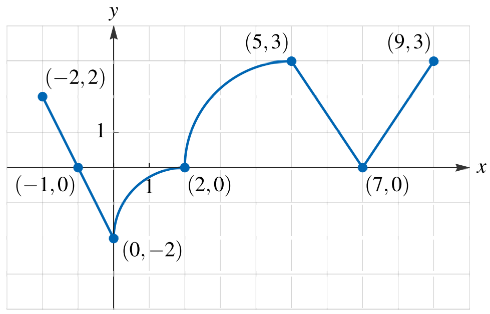
AP Calculus Mock Exam

AB 1

The continuous function f has domain $-2 \le x \le 9$. The graph of f, consisting of three line segments and two quarter circles, is shown in the figure.



Graph of f

Let g be the function defined by $g(x) = \int_0^x f(t)\,dt$ for $-2 \le x \le 9$.

(a) Find the x-coordinate of each critical point of g on the interval $-2 \le x \le 9$.

(b) Classify each critical point from part (a) as the location of a relative minimum, a relative maximum, or neither for g. Justify your answers.

(c) For $-2 \le x \le 9$, on what open intervals is g increasing and concave down? Give a reason for your answer.

(d) Find the value of $g(-1)$. Show the computations that lead to your answer.

(e) Find the value of $g(2)$. Show the computations that lead to your answer.

(f) Find the absolute maximum value of g over the interval $-2 \le x \le 5$.

(g) Find the value of $g''(6)$, or explain why it does not exist.

(h) Must there exist a value of d, for $0 < d < 2$, such that $g'(d)$ is equal to the average rate of change of g over the interval $0 \le x \le 2$? Justify your answer.

(i) Find $\lim_{x \to 0} \dfrac{3x + g(x)}{\sin x}$. Show the computations that lead to your answer.

(j) The function h is defined by $h(x) = x \cdot g(x^2)$. Find $h'(\sqrt{2})$. Show the computations that lead to your answer.

Solution	Scoring
(a) $g'(x) = f(x) = 0 \quad \Rightarrow \quad x = -1,\ 2,\ 7$	1: answer
(b) At $x = -1$, g has a relative maximum because $g'(x) = f(x)$ changes from positive to negative there. At $x = 2$, g has a relative minimum because $g'(x) = f(x)$ changes from negative to positive there. At $x = 7$, g has neither because $g'(x) = f(x)$ does not change sign there.	3: $\begin{cases} 1 : x = -1 \text{ relative maximum with justification} \\ 1 : x = 2 \text{ relative minimum with justification} \\ 1 : x = 7 \text{ neither with justification} \end{cases}$
(c) g is increasing where $g' = f$ is positive. g is concave down where $g' = f$ is decreasing. g is increasing and concave down on the intervals $(-2, -1)$ and $(5, 7)$.	2 : $\begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$
(d) $g(-1) = \int_0^{-1} f(t)\,dt$ $= -\int_{-1}^{0} f(t)\,dt = -\left(-\frac{1}{2}(1)(2)\right) = 1$	1 : answer
(e) $g(2) = \int_0^{2} f(t)\,dt$ $= -\left(2 \cdot 2 - \frac{1}{4} \cdot \pi \cdot 2^2\right) = -(4 - \pi)$	2 : $\begin{cases} 1 : \text{area of quarter circle} \\ 1 : \text{answer} \end{cases}$
(f) The absolute maximum value occurs at an endpoint of the interval or a critical point. Consider a table of values. $x = -2$: $g(x) = 0$ $x = -1$: $g(x) = 1$ $x = 2$: $g(x) = \pi - 4$ $x = 5$: $g(x) = \pi - 4 + \frac{1}{4}\pi 3^2 = \frac{13}{4}\pi - 4$ The absolute maximum value of g is $\frac{13}{4}\pi - 4$.	4: $\begin{cases} 1 : \text{considers } x = -2 \text{ and } x = 5 \\ 1 : \text{considers } x = -1 \text{ and } x = 2 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

Solution	Scoring
(g) $g''(6) = \frac{3-0}{5-7} = -\frac{3}{2}$	1 : answer
(h) $g' = f \quad \Rightarrow \quad g$ is differentiable on $0 < x < 2 \quad \Rightarrow \quad g$ is continuous on $0 \le x \le 2$ Therefore, the Mean Value Theorem can be applied to g on the interval $0 \le x \le 2$ to guarantee that there exists a value of d, for $0 < d < 2$, such that $g'(d)$ equals the average rate of change of g over the interval $0 \le x \le 2$.	2 : { 1 : conditions 1 : conclusion using Mean Value Theorem
(i) $\lim_{x\to 0}(3x + g(x)) = 0$ $\lim_{x\to 0} \sin x = 0$ Therefore the limit $\lim_{x\to 0} \frac{3x+g(x)}{\sin x}$ is in the indeterminate form $\frac{0}{0}$ and L'Hospital's Rule can be applied. $\lim_{x\to 0} \frac{3x+g(x)}{\sin x} = \lim_{x\to 0} \frac{3+g'(x)}{\cos x} = \frac{3+g'(0)}{\cos 0}$ $= \frac{3+f(0)}{\cos 0} = \frac{3+-2}{1} = 1$	3 : { 1 : conditions for L'Hospital's Rule 1 : applies L'Hospital's Rule 1 : answer
(j) $h'(x) = 1 \cdot g(x^2) + x \cdot g'(x^2) \cdot 2x$ $= g(x^2) + 2x^2 f(x^2)$ $h'(\sqrt{2}) = g(2) + 2 \cdot 2 \cdot f(2)$ $= (\pi - 4) + 4 \cdot 0 = \pi - 4$	3 : { 1 : product rule 1 : chain rule 1 : answer