AP Calculus Mock Exam

BC 2

t	0	2	6	8	10	12
$y'(t)$	4	8	−2	3	−1	−5

A particle moves in the coordinate plane with position $(x(t), y(t))$ at time t, where t is measured in seconds and $x(t)$ and $y(t)$ are twice-differentiable functions, both measured in meters.

For all times t, the x-coordinate of the particle's position has derivative $x'(t) = \frac{t}{\sqrt{t^2 + 25}}$. Selected values of $y'(t)$, the derivative of $y(t)$, over the interval $0 \le t \le 12$ seconds are shown in the table.

The position of the particle at time $t = 12$ is $(x(12), y(12)) = (4, -3)$.

(a) Using correct units, find the speed of the particle at time $t = 6$.

(b) Find the exact value of $x(4)$, the x-coordinate of the position of the particle at time $t = 4$.

(c) Using Euler's method, starting at time $t = 12$ with 4 steps of equal size, approximate $y(4)$, the y-coordinate of the position of the particle at $t = 4$.

(d) Find an equation of the line tangent to the path of the particle at time $t = 12$.

(e) Let $r(t)$ be the distance between the particle and the origin $(0, 0)$ at time t. Find $r'(12)$.

(f) Is the particle moving closer or further from the origin at time $t = 12$? Justify your answer.

(g) Given $y''(12) = -2$ and $y'''(12) = 8$, find the third-degree Taylor polynomial approximation for y about $t = 12$.

(h) Suppose that over the time interval $[12, 15]$ the y-coordinate of the position of the particle is the same as the Taylor polynomial approximation found in part (g). Set up but do not evaluate an expression that represents the total distance traveled by the particle over the interval $[12, 15]$.

Solution	Scoring
(a) $x'(6) = \frac{6}{\sqrt{6^2+25}} = \frac{6}{\sqrt{61}}$ and $y'(6) = -2$ speed $= \sqrt{[x'(6)]^2 + [y'(6)]^2}$ $= \sqrt{\left(\frac{6}{\sqrt{61}}\right)^2 + (-2)^2}$ $= \sqrt{\frac{36}{61} + 4} = 2\sqrt{\frac{70}{61}}$ m/s	2: $\begin{cases} 1 : \text{expression for speed} \\ 1 : \text{answer with units} \end{cases}$
(b) $x(4) = x(12) - \int_4^{12} x'(t)\,dt$ $= (4) - \int_4^{12} \frac{t}{\sqrt{t^2+25}}\,dt$ $= (4) - \left[\sqrt{t^2+25}\right]_4^{12}$ $= (4) - [\sqrt{169} - \sqrt{41}] = \sqrt{41} - 9$	3: $\begin{cases} 1 : \text{integral} \\ 1 : \text{uses } x(12) \\ 1 : \text{answer} \end{cases}$
(c) $y(10) \approx y(12) + (-2) \cdot y'(12) = -3 + (-2)(-5) = 7$ $y(8) \approx y(10) + (-2) \cdot y'(10) = 7 + (-2)(-1) = 9$ $y(6) \approx y(8) + (-2) \cdot y'(8) = 9 + (-2)(3) = 3$ $y(4) \approx y(6) + (-2)y'(6) = 3 + (-2)(-2) = 7$	2: $\begin{cases} 1 : \text{First step in Euler's method} \\ 1 : \text{answer} \end{cases}$
(d) $\left.\frac{dy}{dx}\right\|_{t=12} = \frac{y'(12)}{x'(12)} = \frac{-5}{\frac{12}{\sqrt{12^2+25}}} = -\frac{65}{12}$ An equation of the tangent line at the position $(x(12), y(12)) = (4, -3)$ is $y = -\frac{65}{12}(x-4) - 3$	2: $\begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line equation} \end{cases}$

Solution	Scoring
(e) $r(t) = \sqrt{x(t)^2 + y(t)^2}$ $r'(t) = \frac{1}{2}(x(t)^2 + y(t)^2)^{-1/2}(2x(t)x'(t) + 2y(t)y'(t))$ $= \frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{x(t)^2 + y(t)^2}}$ $r'(12) = \frac{x(12)x'(12) + y(12)y'(12)}{\sqrt{x(12)^2 + y(12)^2}}$ $= \frac{(4)\left(\frac{12}{13}\right) + (-3)(-5)}{\sqrt{4^2 + (-3)^2}}$ $= \frac{243}{65}$	3: $\begin{cases} 1 : \text{expression for } r(t) \\ 1 : \text{expression for } r'(t) \\ 1 : \text{answer} \end{cases}$
(f) $r'(12) = \frac{243}{65} > 0$ The distance r to the origin is increasing at time $t = 12$. Therefore the particle is moving away from the origin at time $t = 12$ seconds.	1 : answer with reason
(g) Let $g(t)$ be the third degree Taylor polynomial for y at $t = 12$. $g(t)$ $= y(12) + y'(12)(t-12) + \frac{y''(12)}{2!}(t-12)^2 + \frac{y'''(12)}{3!}(t-12)^3$ $= -3 + (-5)(t-12) + \frac{-2}{2}(t-12)^2 + \frac{8}{6}(t-12)^3$ $= -3 - 5(t-12) - (t-12)^2 + \frac{4}{3}(t-12)^3$	2: $\begin{cases} 1 : \text{two terms of the Taylor polynomial} \\ 1 : \text{remaining terms} \end{cases}$
(h) For $12 \le t \le 15$, $y(t) = g(t) = -3 - 5(t-12) - (t-12)^2 + \frac{4}{3}(t-12)^3$ $x'(t) = \frac{t}{\sqrt{t^2+25}}$ $y'(t) = g'(t) = -5 - 2(t-12) + 4(t-12)^2$ The distance traveled by the particle over the time interval $[12, 15]$ is $\int_{12}^{15} \sqrt{\left[\frac{t}{\sqrt{t^2+25}}\right]^2 + \left[-5 - 2(t-12) + 4(t-12)^2\right]^2}\, dt$	2: $\begin{cases} 1 : \text{expression for } y'(t) \\ 1 : \text{expression for distance traveled} \end{cases}$