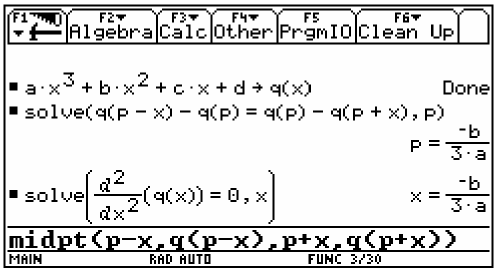
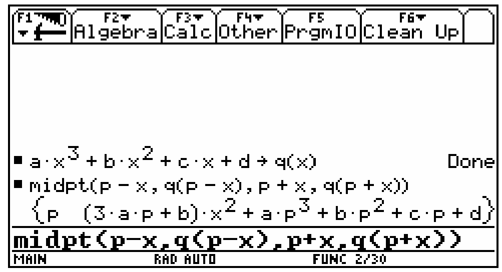
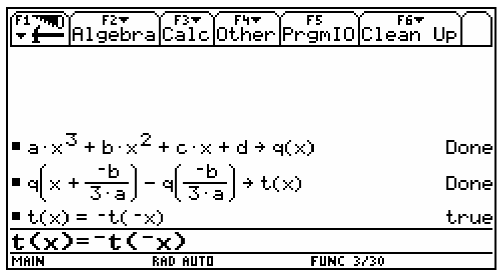
**Problems for CAS Solution**
Presented by
Lin McMullin

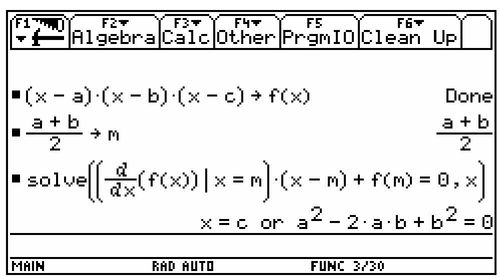
1. Prove that the graph of every cubic polynomial has a point of symmetry (or the graph is symmetric to its point of inflection).



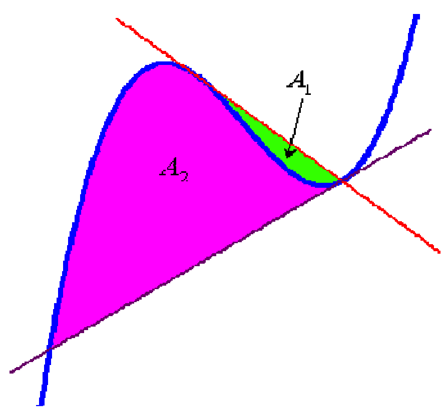




2. Prove that the tangent line drawn to a cubic polynomial at the point where $x$ = average of two of its roots, intersects the polynomial on the $x$-axis at the third root.




Beside what we were expecting, what does the last line tell us?

3. Draw a tangent line at any point, other than the point of inflection of a cubic polynomial. This tangent will intersect the cubic at a second point; draw a tangent line at this second point. The second tangent will intersect the cubic at a third point. Let $A_1$ be the area of the region between the first tangent line and the cubic and let $A_2$ be the area of the region between the cubic and the second tangent line. A general graph is given below. The interesting result is that the ratio $A_2 : A_1$ is constant.
(A) Find the ratio $A_2$: $A_1$.
(B) Prove that the ratio is constant.



Suggested by *Algebra in Motion* by Audrey Weeks at [www.calculusinmotion.com](http://www.calculusinmotion.com)
(A) 16:1;   (B): CAS solution:

F1 | F2 Algebra | F3 Calc | F4 Other | F5 PrgmIO | F6 Clean Up

1 $a \cdot x^3 + b \cdot x^2 + c \cdot x + d \rightarrow f(x)$ Done

2 $\frac{d}{dx}(f(x)) \mid x = m$ $\quad 3 \cdot a \cdot m^2 + 2 \cdot b \cdot m + c$

3 $(3 \cdot a \cdot m^2 + 2 \cdot b \cdot m + c) \cdot (x - m) + f(m) \rightarrow t1(x)$ Done

4 $\text{solve}(f(x) = t1(x), x)$ $\quad x = \frac{-(2 \cdot a \cdot m + b)}{a} \text{ or } x = m$

5 $\frac{-(2 \cdot a \cdot m + b)}{a} \rightarrow n$ $\quad \frac{-(2 \cdot a \cdot m + b)}{a}$

6 $\frac{d}{dx}(f(x)) \mid x = n$ $\quad \frac{12 \cdot a^2 \cdot m^2 + a \cdot (8 \cdot b \cdot m + c) + b^2}{a}$

7 $\frac{12 \cdot a^2 \cdot m^2 + a \cdot (8 \cdot b \cdot m + c) + b^2}{a} \cdot (x - n) + f(n) \rightarrow t2(x)$ Done

8 $\text{solve}(f(x) = t2(x), x)$ $\quad x = \frac{4 \cdot a \cdot m + b}{a} \text{ or } x = \frac{-(2 \cdot a \cdot m + b)}{a}$

9 $\frac{4 \cdot a \cdot m + b}{a} \rightarrow p$ $\quad \frac{4 \cdot a \cdot m + b}{a}$

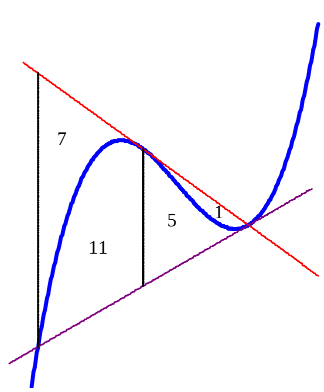
10 $\int_p^n (f(x) - t2(x))\,dx$ $\quad \frac{4 \cdot (81 \cdot a^4 \cdot m^4 + 108 \cdot a^3 \cdot b \cdot m^3 + 54 \cdot a^2 \cdot b^2 \cdot m^2 + 12 \cdot a \cdot b^3 \cdot m + b^4)}{3 \cdot a^3}$

11 $\int_m^n (t1(x) - f(x))\,dx$ $\quad \frac{81 \cdot a^4 \cdot m^4 + 108 \cdot a^3 \cdot b \cdot m^3 + 54 \cdot a^2 \cdot b^2 \cdot m^2 + 12 \cdot a \cdot b^3 \cdot m + b^4}{12 \cdot a^3}$

12 $\frac{\frac{4 \cdot (81 \cdot a^4 \cdot m^4 + 108 \cdot a^3 \cdot b \cdot m^3 + 54 \cdot a^2 \cdot b^2 \cdot m^2 + 12 \cdot a \cdot b^3 \cdot m + b^4)}{3 \cdot a^3}}{\frac{81 \cdot a^4 \cdot m^4 + 108 \cdot a^3 \cdot b \cdot m^3 + 54 \cdot a^2 \cdot b^2 \cdot m^2 + 12 \cdot a \cdot b^3 \cdot m + b^4}{12 \cdot a^3}}$ $\quad 16$

MAIN RAD AUTO FUNC 12/30

And some other Ratios:

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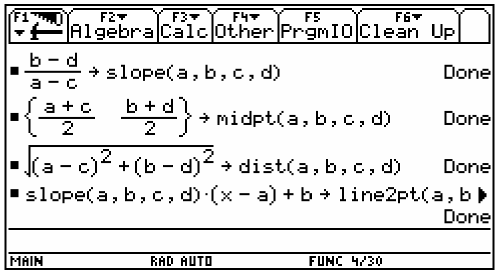
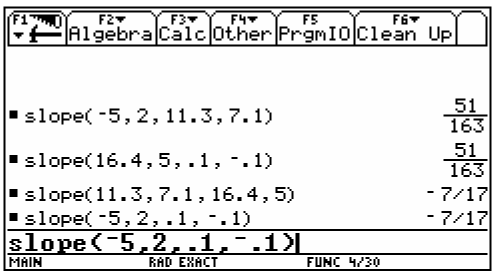
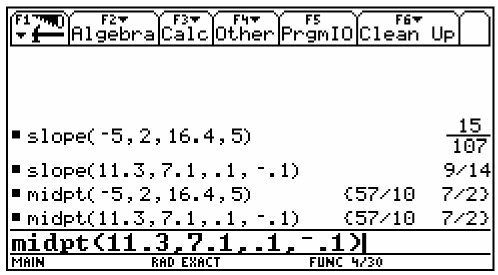
Analytic Geometry:

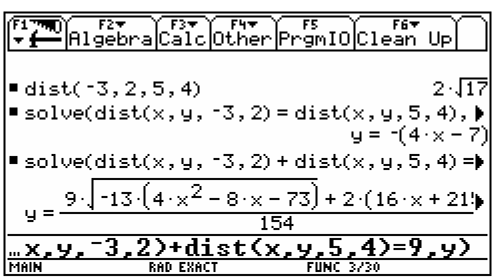


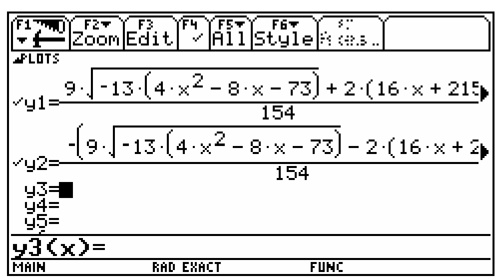

4. Given the quadrilateral with vertices $A(-5,2)$, $B(11.3,7.1)$, $C(16.4,5.0)$ and $D(0.1,-0.1)$
   (a) Show that *ABCD* is a parallelogram.
   (b) Are the diagonals perpendicular? Show how you know.
   (c) Show that the diagonals bisect each other.

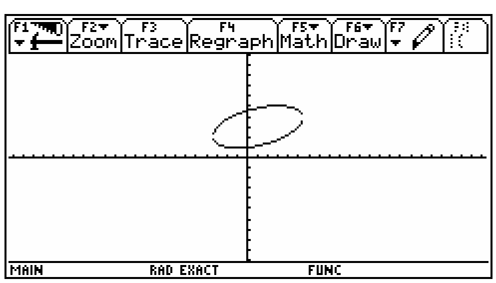


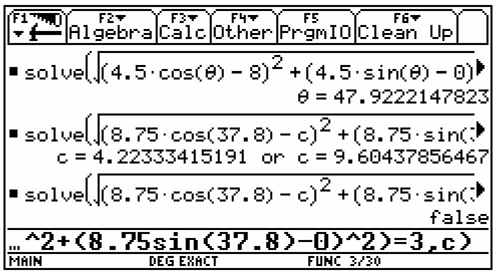

5. Given the points $A(-3,2)$ and $B(5,4)$
   a. Find the length *AB*.
   b. Write an equation of the perpendicular bisector of $\overline{AB}$.
   c. Write an equation of the set of points (*x*, *y*) such that the sum of the distances from (*x*, *y*) to *A* and *B* is 9.
   d. Graph the locus found in part (c).

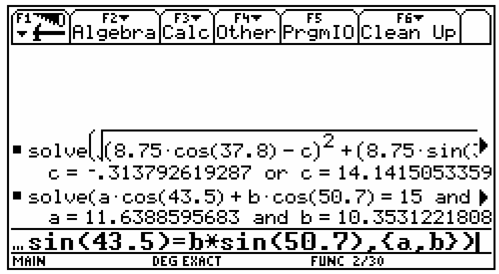


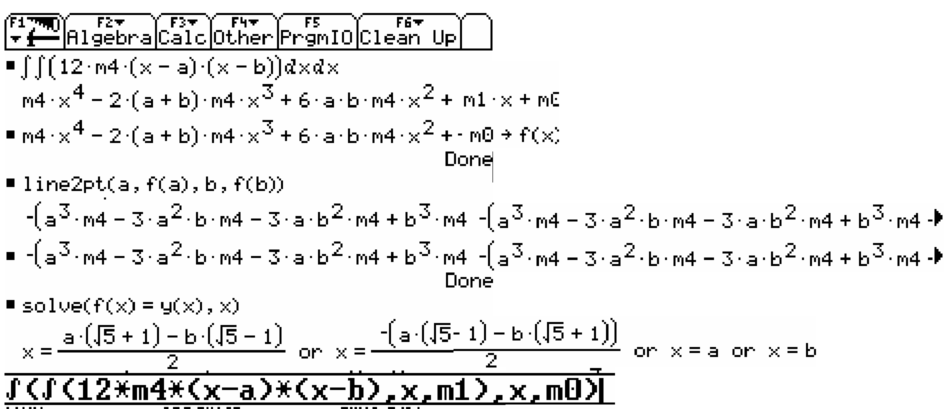
6. Trigonometry.
    a. SSS: A triangle has sides of 4.5, 6 and 8. Find the measure of the angle opposite the side of 6.
    b. SSA: In triangle ABC, angle A = 37.8°, side $b$ = 8.75 and side $a$ = 6. Find the measure of length of side AB = $c$.
    c. SSA: In triangle ABC, angle A = 37.8°, side $b$ = 8.75 and side $a$ = 3. Find the measure of length of side AB = $c$.
    d. SSA: In triangle ABC, angle A = 37.8°, side $b$ = 8.75 and side $a$ = 9. Find the measure of length of side AB = $c$.
    e. ASA : In triangle ABC, angle A = 50.7°, angle B = 43.5° and AB = 15. Find the lengths of the other 2 sides.




7. Where else does the line through the points of inflection of a 4th degree polynomial intersect the polynomial?

    (Note: First line's entry format is shown at the bottom; this shows how to enter, + C, the constant of integration, here first $m_1$ then $m_0$,)




    Why is the solution so unexpected?

8. How is doing math with a CAS different than do math without a CAS?

- CAS removes the necessity to do algemetic, so we can concentrate on the mathematics.
- Knowing how to use the CAS allows you to improve the CAS by adding routines you need to do standard problems the “usual” way.
- New approaches for doing problems appear once you stop worrying about the algemetic.
- ‘Go for the equation’ – since the CAS can solve (in closed form or not) almost any equation, you needn’t worry about how difficult it is to solve.
- A CAS will simplify just about any expression so we don’t have to avoid complicated expressions – in fact we can complicate what we have, if that makes the flow of work easier.
- One still needs to know the mathematics to do the set-up and to interpret the answers.

9. What are the implications for teaching?

   Good CAS use is a new skill, not just a new tool that students must be taught and encouraged to learn.

   To do this we will need

- A willingness to accept new ways of doing problems
- A willingness to accept showing a different kind of work
- A change in the meaning of “simplify.”
- A good source of (better) problems for students to attempt.

# SOUNDOFF! and DISCUSS WITH YOUR COLLEAGUES

Lin McMullin

# Algemetic

Item: A few years ago I ran across an ad for a well-known symbol-manipulation software package. The left-hand page extolled the virtues of the software, and the right-hand page listed all the things that it could do. The list had three columns, running about two-thirds the length of the page. It went from simple addition through matrix arithmetic and multiple integration. I estimate that a typical student takes at least six years in high school and college to learn how to do what was listed on that two-thirds of a page.

The "Rule of Four" is quickly becoming the guiding idea in mathematics education. The Rule of Four states that mathematics should be studied and taught from the (1) numerical; (2) analytical, that is, meaning symbols and equations; (3) geometric, or graphical; and (4) verbal perspectives. In the past, a graph or a number was the result of a long process of working with symbols. The advent of technology, particularly graphing calculators and inexpensive computer programs, has changed this process. The graphs and numbers have become the starting point, not the end result. Such national tests as the Advanced Placement calculus examinations expect students to be able to work problems and understand concepts without having symbols or equations with which to work.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I think that this development is good.

The use of this technology is controversial because symbol manipulation has been pushed out of its former prominent place in the curriculum. Things continue to change: computer software and calculators can do the symbol manipulation—the factoring, the solving, the rationalizing—in addition to the graphing. These calculators and software packages are available today; make no mistake about it, they will not go away. The high school mathematics curriculum and pedagogy have to change.

I think that this development, too, is good.

This development is a good one because it will allow students to go deeper into the subject. It will allow them to be more creative. It will encourage their curiosity. It will accomplish these ends primarily by removing the tediousness associated with symbol manipulation from algebra through calculus.

Teachers need to decide exactly how much and what kinds of symbol-manipulating skills are required. Symbol manipulation is what makes mathematics work. It is neither mindless nor unnecessary. Students need to be able to use, interpret, read, write, decode, and otherwise handle symbols. You really cannot do mathematics without that ability, but mathematics is a lot more than symbols.

To many people, mathematics, particularly algebra, IS symbol manipulation. Too many—far too many—students think that mathematics is ONLY symbol manipulation. Why? The reason is obvious: It's what is taught in high school. It's what is on the homework. It's what is on the quizzes and the tests and the SATs. The student who can manipulate symbols gets a good grade. As mathematics teachers, that's what we spend most of our time teaching.

People believe that all sorts of horrible things will result if less time is spent teaching students how to do algebra. What they mean by *algebra* is how to manipulate symbols—factor, rationalize denominators, solve equations, and so on.

I do not agree.

Machines can do the manipulation: let them!

Allow me to coin a word. The word is *algemetic;* it rhymes with *arithmetic. Algemetic* means all the symbol manipulating that the Casio Algebra FX 2.0, HP38, HP48, TI-89, TI-92, Derive, Mathematica, Maple, MathCad, and the like, can do. Algemetic is not all that algebra is, but it is what students spend years learning.

Instruction must include having students manipulate variables with pencil and paper. The goal should not be to get students to be great "algemeticians" but rather to enable them to know the algemetic that can and should be done to solve a particular problem or investigate a particular situ-

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ation. To know when, where, and what is appropriate is more important than the ability to do it by hand. In other words, students have to know what to do, why they are doing it, and when to do it, but they don't have to be good at it.

Before jumping on that statement, consider the following:

- You balance your checkbook with a calculator, do you not? Admit it. Even though you probably have several degrees in mathematics and can do arithmetic more accurately and quicker than most people can, you still use a calculator when it really counts—after all, it's your money.
- When scientific calculators first came out, I was still required for too many years to teach how to compute with logarithms. I can still see the students sitting there and adding up logarithms with their scientific calculators.
- Mathematics in the real world is done with technology. Very few, if any, working mathematicians and engineers do complicated symbol manipulation by hand. In today's world, not instructing students on the use of technology is doing them a disservice.

People still need to understand what the machines are doing, what the symbols mean, how they apply to the problem under consideration, and so on. People need more than ever to understand mathematics, particularly algebra and calculus. But when it comes to algemetic, what is the problem with letting the machines do it? Machines are better, faster, and more accurate than people are. If someone needs to do a computation, solve an equation, factor, rationalize a denominator, find a derivative, multiply matrices, or even add a few integers, what is so bad about pushing a few buttons?

Starting in kindergarten, students should learn to apply the Rule of Four. Put the emphasis on mathematics, not algemetic. Concentrate on the ideas. Concentrate on the numerical, analytic, graphical, and verbal relationships among the ideas. Concentrate on communicating those ideas and relationships.

Use technology not as a quick way to the answer but as a tool for investigation. Use technology to delve deeper into the problem, not to drive straight through to the answer. Use it as a way to keep focused on the mathematics by avoiding the algemetic. Use the technology correctly and to its fullest extent, not just to add logarithms.

These changes, like all change, will cause conflict. The beliefs and expectations of various groups will be challenged. Those who learned "the old way" think that it was the best way. Those who teach "the old way" will resist change, partly because of their success with it and partly because of their own reluctance to learn the new techniques. Yet students' future employers will expect them to be able to use and understand the newest technology and to be able to learn the emerging technology. Technology belongs in the mathematics classroom from the first day.

Back to the software ad: six years of work reduced to less than a page! Think of all that time being used to really get into mathematics—all that time to investigate, solve problems, think, discuss, and write about what is happening—sure beats factoring $x^2 - 4$ for the hundredth time.

**Discuss:** What skills should be taught by hand, and why?

What successful activities have CAS use supported in your classroom?

What professional development activities would be necessary if your school decided to make extensive use of CASs?