Using Rolle's Theorem To Inscribe An Ellipse With Maximum Area In A Triangle

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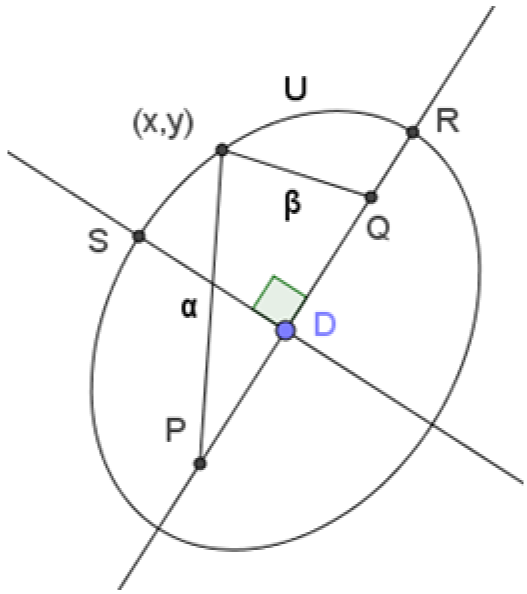
Definition: We say that a subset U of R^2 is an ellipse if and only if there are points P and Q in R^2, and a scalar $L > |P - Q|$, so that U is the set of points (x, y) in R^2 satisfying

$$|(x,y) - P| + |(x,y) - Q| = L$$



The points P and Q in the definition are referred to as the foci of the ellipse, and the center of the ellipse is given by $D = \frac{1}{2}(P + Q)$ (the midpoint of the line segment $\overline{PQ}$). Also, $\alpha + \beta = L$.

Theorem: The subset U in the xy plane is an ellipse with center $D = (x_0, y_0)$ if and only if there are scalars a, b and c so that $a > 0$, $ab - c^2 > 0$, and U is the set of points (x, y) solving the equation

$$a(x - x_0)^2 + b(y - y_0)^2 + 2c(x - x_0)(y - y_0) = 1$$

The ElliptiGo 11R Bicycle –

ELLIPTIGO

http://www.elliptigo.com/

Magnified view.


ELLIPTIGO

Profile shot with graphic overlay.

Geogebra Demo

That was a big fail, but look at this demo?
Does it produce an ellipse?

Geogebra Demo – Perp Jig

Here is another.
Does it produce an ellipse?

Geogebra Demo – NonPerp Jig

OK. Enough… We need some facts.

Suppose a, b, c, d are real numbers and $ad - bc \neq 0$. If S is a subset of the xy plane, then we define the action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on S as the set

$$\left\{(u, v) \;\middle|\; \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \text{ for some } (x, y) \in S\right\}$$

Exercise: Give the action of $\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$ on the line segment connecting the points $(-1,2)$ and $(2,-3)$. Also, find the action of $\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$ on the midpoint of this line segment, and comment on any observations.

In General: The action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on a line segment is a line segment, and the endpoints of the new line segment are the result of the action of $\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$ on the endpoints of the original line segment. In addition, the midpoint of the original line segment gets mapped to the midpoint of the new line segment.

Exercise: Give the action of $\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$ on the triangle with vertices $(-1,2)$, $(2,-3)$ and $(4,1)$. Also, compare the areas.

In General: The action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on a triangle T_1 is a triangle T_2, and

$$\text{area}(T_2) = |ad - bc|\text{area}(T_1)$$

Exercise: Give the action of $\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$ on the set consisting of the parabola $y = x^2$ and its tangent line at $x = 1$. Graph the results.

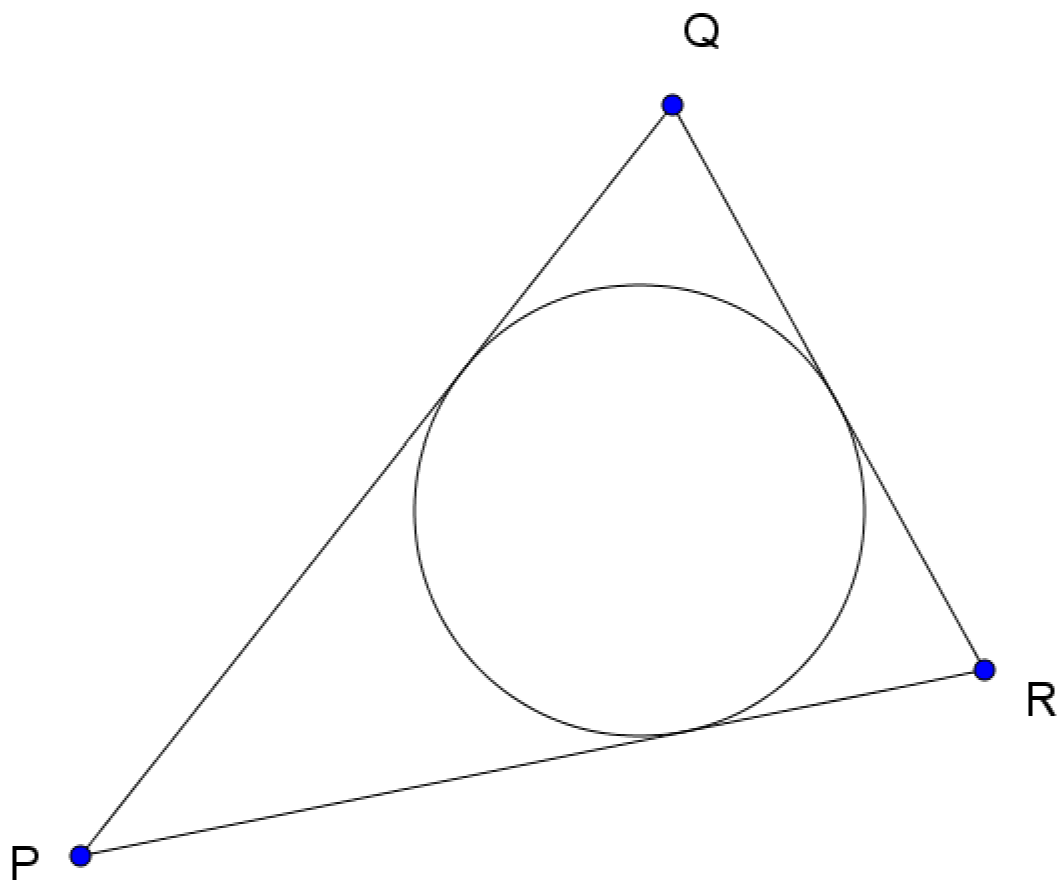
In General: The action of a matrix on a set preserves tangent intersections of curves in the set.

More Fun Facts: The action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on an ellipse E_1 is an ellipse E_2, and

$$\text{area}(E_2) = |ad - bc|\text{area}(E_1)$$

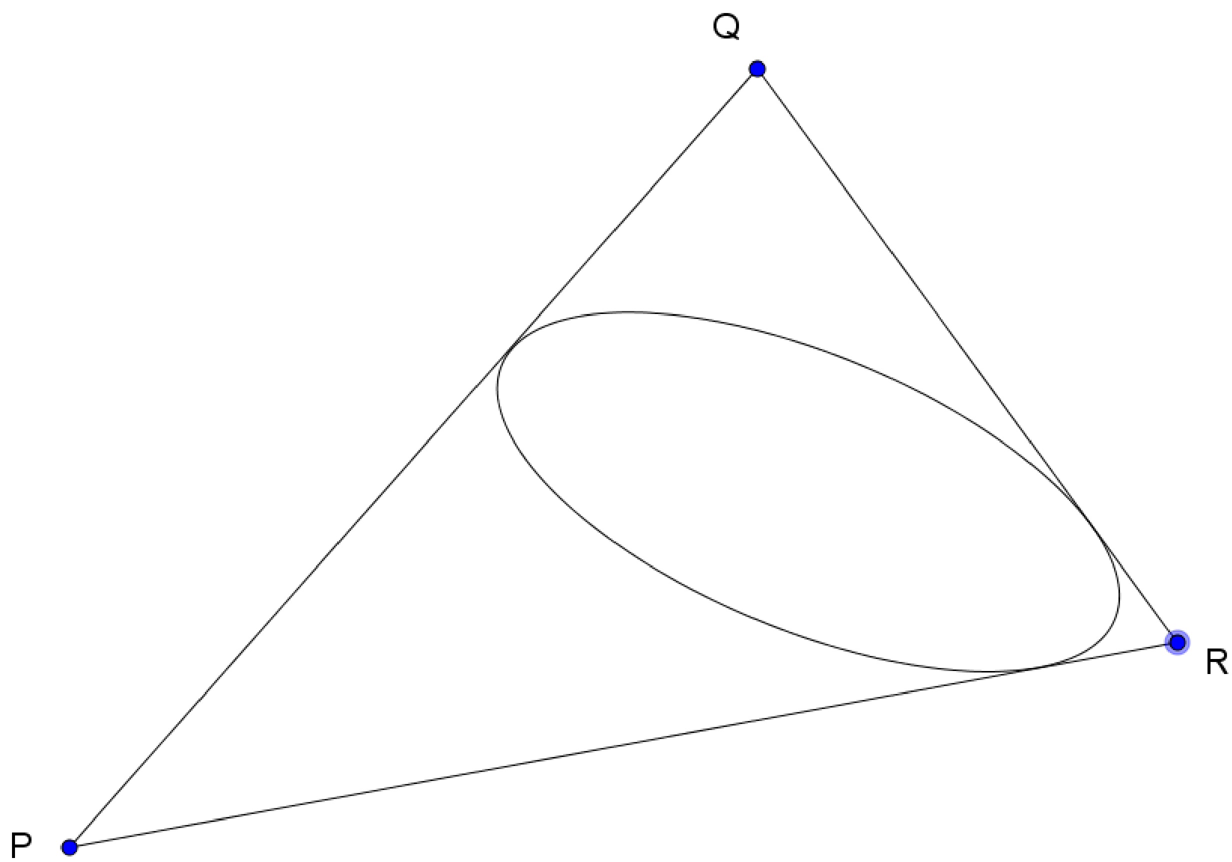
A Simple Minimization Problem

Find a triangle of minimum area for which a given circle of is an incircle.



Our Featured Maximization Problem!!

Find an ellipse of maximum area that is inscribed in a given triangle.



Step 1: Note that maximizing the area of the ellipse is equivalent to maximizing

$$\frac{\text{area}(ellipse)}{\text{area}(triangle)}$$

Because the triangle is fixed.

Step 2: Believe there is an ellipse that has maximum area. Now, take the given triangle and the ellipse (that you don't have) and act a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on the combination so that the ellipse (that you don't have) becomes a circle. **What is**

$$\frac{\text{area}(circle)}{\text{area}(new\ triangle)}\ ?$$

Step 3: How does this help us solve our maximization problem? Hint: See the earlier minimization problem.

Example: Find an ellipse of maximum area that is inscribed in the triangle with vertices $(-1,2)$, $(2,-3)$ and $(4,1)$.

A Much Simpler Solution
and Comments on Rolle's Theorem

Example: Find an ellipse of maximum area that is inscribed in the triangle with vertices $(-1,2)$, $(2,-3)$ and $(4,1)$.

Step 1: Form the function

$$f(z) = (z - (-1 + 2i))(z - (2 - 3i))(z - (4 + i))$$

Step 2: Find the roots of $f'(z)$.

Step 3: Find the roots of $f''(z)$.

Step 4: How is this related to the solution?

Terms: Steiner Inellipse, Marden's Theorem