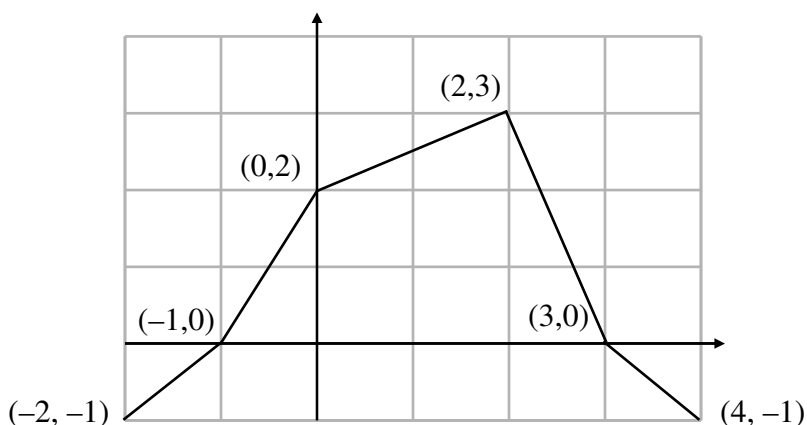


The Detective's Hat Function

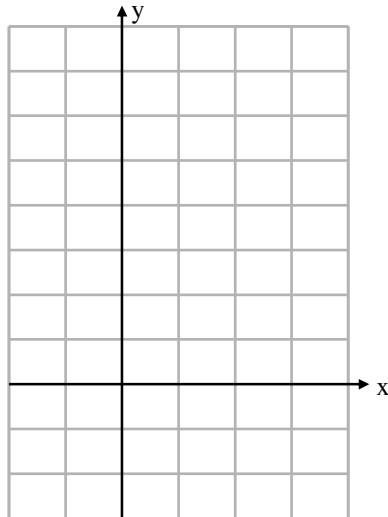


The graph of the function f shown above is a piecewise continuous function defined on $[-2, 4]$. The graph of f consists of five line segments.

Let g be the function given by $g(x) = \int_0^x f(t) dt$.

- Find each of the following.
 - $g(-2)$
 - $g(-1)$
 - $g(0)$
 - $g(2)$
 - $g(3)$
 - $g(4)$
- Explain the procedure you followed to answer question 1.
- Find each of the following.
 - $g'(-1)$
 - $g'(0)$
 - $g'(2)$
 - $g'(4)$
- Explain the procedure you followed to answer question 3.
- Explain why g must be a continuous function on $[-2, 4]$.
- Write the equation for $g'(x)$ on the interval $[0, 2]$.
- Write the equation for the line tangent to g at $x = 1$. Justify your answer.
- Does $g''(0)$ exist? Explain your reasoning.
- Will a point of inflection for g exist when $x = 0$? Explain your reasoning.
- For what values of x in the open interval $(-2, 4)$ is g increasing? Explain your reasoning.
- For what values of x in the open interval $(-2, 4)$ is g decreasing?

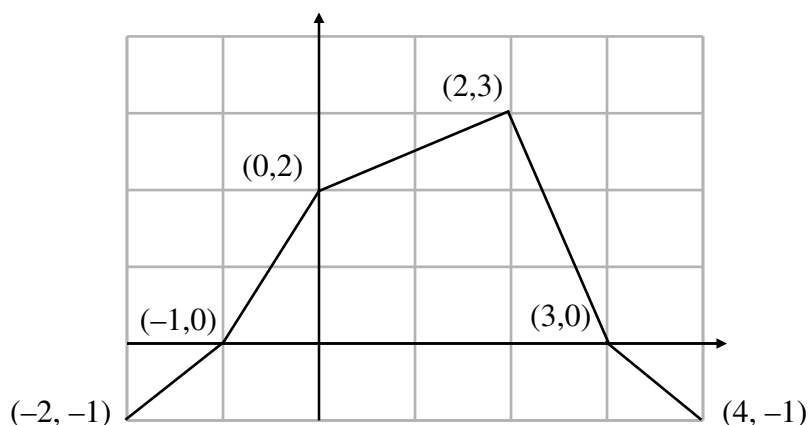
12. For what values of x in the open interval $(-2, 4)$ is g concave up? Explain your reasoning.
13. For what values of x in the open interval $(-2, 4)$ is g concave down?
14. Find the maximum and the minimum values of g on the closed interval $[-2, 4]$. Justify your answers.
15. On the axes provided, sketch the graph of function g on the closed interval $[-2, 4]$.



For questions 15 – 17, let h be the function given by $h(x) = \int_{-2}^x f(t) dt$.

16. Find each of the following.
 - (a) $h(-2)$
 - (b) $h(-1)$
 - (c) $h(0)$
 - (d) $h(2)$
 - (e) $h(3)$
 - (f) $h(4)$
17. Find the following and explain your reasoning.
 - (a) $h'(-1)$
 - (b) $h'(0)$
 - (c) $h'(2)$
 - (d) $h'(4)$
18. $g(x) - h(x) = k$, where k is a constant. Find the value of k and explain your reasoning.
19. If $w(x) = \int_3^x f(t) dt$, find $w(0)$.
20. $w(x)$ can also be defined as $w(x) = r + \int_0^x f(t) dt$ where r is a constant. What is the value of r ?

The Detective's Hat Function Answers



The graph of the function f shown above is a piecewise continuous function defined on $[-2, 4]$. The graph of f consists of five line segments.

Let g be the function given by $g(x) = \int_0^x f(t) dt$.

1. Find each of the following.

(a) $g(-2) = -\frac{1}{2}$ (b) $g(-1) = -1$ (c) $g(0) = 0$ (d) $g(2) = 5$
(e) $g(3) = 6.5$ (f) $g(4) = 6$

2. Explain the procedure you followed to answer question 1.

Since the velocity is made up of line segments, the definite integrals can be calculated using geometry. $\int_0^{-2} f(t) dt$ is the sum of two triangles.

$$\frac{1}{2}(-1)(-1) + \frac{1}{2}(-1)(2) = -\frac{1}{2}. \text{ The other values are calculated in a similar fashion.}$$

3. Find each of the following.

(a) $g'(-1) = 0$ (b) $g'(0) = 2$ (c) $g'(2) = 3$ (d) $g'(4) = -1$

4. Explain the procedure you followed to answer question 3.

Since the derivative of g is equal to $f(x)$, the values of the derivative are the same as the values of f .

5. Explain why g must be a continuous function on $[-2, 4]$.

Since $g'(x)$ exists on the interval $[-2, 4]$ g must be continuous on the interval.

6. Write the equation for $g'(x)$ on the interval $[0, 2]$.

$$g'(x) = \frac{1}{2}x + 2$$

7. Write the equation for the line tangent to g at $x = 1$. Justify your answer.

$$g(1) = \frac{9}{4}, \quad g'(1) = \frac{5}{2}, \quad y = \frac{5}{2}(x-1) + \frac{9}{4}$$

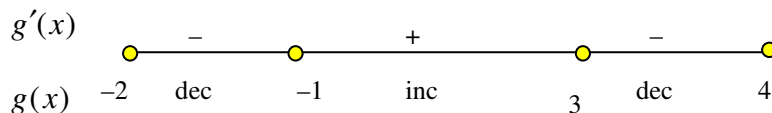
8. Does $g''(0)$ exist? Explain your reasoning.

No, g' has a cusp at 0; therefore, $g''(0)$ will not exist. (Or show that the right and left limits of g'' are not the same.)

9. Will a point of inflection for g exist when $x = 0$? Explain your reasoning.

Since g' is increasing on both sides of 0, there will not be a point of inflection at $x = 0$.

10. For what values of x in the open interval $(-2, 4)$ is g increasing? Explain your reasoning.



g is increasing for $-1 < x < 3$ where $g' > 0$.

11. For what values of x in the open interval $(-2, 4)$ is g decreasing?

g is decreasing for $-2 < x < -1$ and again when $3 < x < 4$ where $g' < 0$.

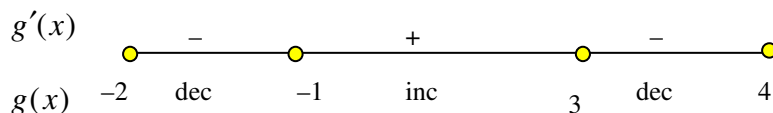
12. For what values of x in the open interval $(-2, 4)$ is g concave up? Explain your reasoning.

g will be concave up when g' is increasing and where $g''(x)$ exists and is not equal to 0. g is concave up for $(-2, -1)$, $(-1, 0)$, and $(0, 2)$. (Note: A function is concave up if the second derivative is positive and concave down when the second derivative is negative. No concavity exists when the second derivative is 0. This is why a straight line is neither concave up or down.)

13. For what values of x in the open interval $(-2, 4)$ is g concave down?

g will be concave down when g' is decreasing and where $g''(x)$ exists and is not equal to 0. g is concave down for $(2, 3)$, and $(3, 4)$.

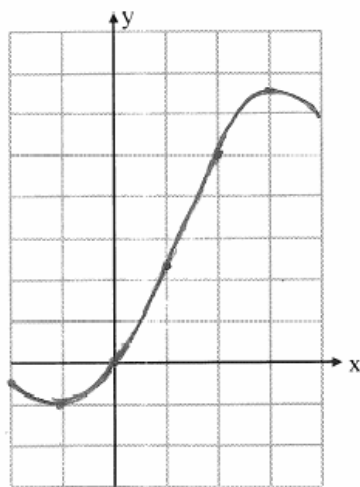
14. Find the maximum and the minimum values of g on the closed interval $[-2, 4]$. Justify your answers.



The maximum value must occur at $x = -2$ or at $x = 3$. $g(-2) = -\frac{1}{2}$ and $g(3) = 6.5$; therefore, the maximum value of g is 6.5.

The minimum value must occur at $x = -1$ or at $x = 4$. $g(-1) = -1$ and $g(4) = 6$; therefore, the minimum value of g is -1 .

15. On the axes provided, sketch the graph of function g on the closed interval $[-2, 4]$.



(This function is concave up on the intervals $(-2, -1)$, $(-1, 0)$, and $(0, 2)$; however, on the interval $(0, 2)$ the curvature is so slight that the function appears to be linear) The slope of the tangent line at $x = 0$ is 2.

For questions 15 – 17, let h be the function given by $h(x) = \int_{-2}^x f(t) dt$.

(a) $h(-2) = 0$ (b) $h(-1) = -\frac{1}{2}$ (c) $h(0) = \frac{1}{2}$ (d) $h(2) = \frac{11}{2}$

(e) $h(3) = 7$ (f) $h(4) = \frac{13}{2}$

17. Find the following and explain your reasoning.

(a) $h'(-1) = -2$ (b) $h'(0) = 2$ (c) $h'(2) = 3$ (d) $h'(4) = -1$

18. $g(x) - h(x) = k$, where k is a constant. Find the value of k and explain your reasoning.

Since g and h have the same derivative, they can differ at most by a constant. That constant is $\frac{1}{2}$. Or

$$\begin{aligned} k &= \int_0^x f(t) dt - \int_{-2}^x f(t) dt \\ &= \int_0^x f(t) dt + \int_x^{-2} f(t) dt \\ &= \int_0^{-2} f(t) dt \\ &= -\int_{-2}^0 f(t) dt \\ k &= -\frac{1}{2} \end{aligned}$$

19. If $w(x) = \int_3^x f(t) dt$, find $w(0)$. $\int_3^0 f(t) dt = -\int_0^3 f(t) dt = -6\frac{1}{2}$

20. $w(x)$ can also be defined as $w(x) = r + \int_0^x f(t) dt$ where r is a constant. What is the value of r ? $-6\frac{1}{2}$