From Difference Quotient to Derivative

Given a function $y = f(x)$ that is defined on a closed interval $[a,b]$.

The difference quotient $\frac{f(b)-f(a)}{b-a}$ is called the ***average rate of change of*** f ***with respect to*** x over the interval $[a,b]$.

> Note: This difference quotient could also be expressed as $\frac{f(x_o+h)-f(x_o)}{h}$ and would thereby be the average rate of change of f with respect to x over the interval $[x_o, x_o+h]$

When seeking a "good" linear model for f at $(a, f(a))$, it is a difference quotient involving $(a, f(a))$ that is frequently used.

In Calculus, we wrestle with the concept of instantaneous rate of change…an apparent oxymoron. The instantaneous rate of change of f with respect to x **<u>at</u>** $(a, f(a))$ would, at first, seem to be $\frac{f(a)-f(a)}{a-a}$ - a clearly undefined, and hence meaningless, expression. In fact, for centuries, mathematicians have understood what they were trying to capture when discussing instantaneous rate of change, but they have had tremendous difficulty defining the concept precisely. When the concept of derivative (instantaneous rate of change) was addressed by Leibniz, Berkeley, and their contemporaries of the late 17th and early 18th centuries, each person used his or her own unique language and notation to describe the elusive concept. Leibniz thought of the derivative as a difference quotient involving infinitely small (but non-zero) quantities. Berkeley gleefully pronounced the derivative to be a difference quotient involving the "ghosts of departed quantities". It was not until the middle of the 19th century that Weierstrass and Cauchy moved mathematics forward by replacing infinitesimals with the concept of limit.

We now define the **instantaneous rate of change of *f* with respect to *x* at *x*=*a*** to be $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$, if this limit exists. Note: This concept can be described using a slight variation of notation to be $\lim_{h\to 0}\frac{f(x_o+h)-f(x_o)}{h}$, the **instantaneous rate of change of *f* with respect to *x* at** $x=x_o$.

Checking for Clarity

1. Given that $f(x) = 3x^2 + 4$.
   a. Use the limiting value of an appropriate difference quotient to find the instantaneous rate of change of *f* with respect to *x* at *x* = -2.
   b. Write an equation for the best linear model for *f* at *x* = -2.
   c. Use the limiting value of an appropriate difference quotient to find the instantaneous rate of change of *f* with respect to *x* at $x = x_o$.

2. Given that $g(x) = 3x^2 - \pi$.
   a. Use the limiting value of an appropriate difference quotient to find $g'(-2)$.
   b. Use the limiting value of an appropriate difference quotient to find $g'(x_o)$
   c. Write a brief paragraph to explain ***how*** and ***why*** the results of questions 1 a,c and 2 a,b are related.

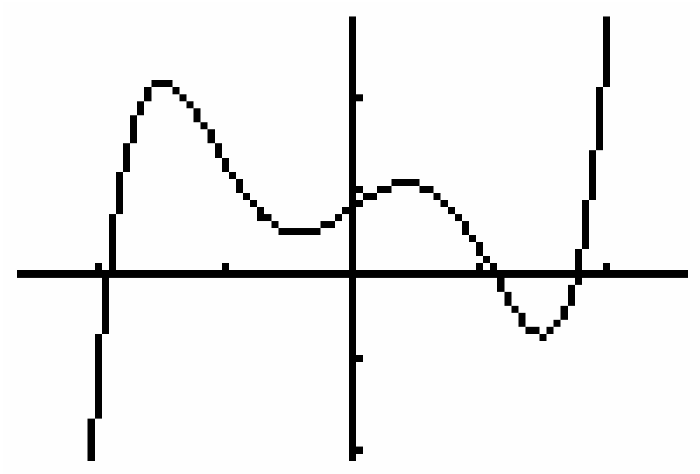
3. Given that $h(x) = \sqrt{x^2 - 3}$.
   a. Use the limiting value of an appropriate difference quotient to find $h'(1)$
   b. Use the limiting value of an appropriate difference quotient to find $h'(a)$.
   c. Use your result from part b to find an approximation for $h(2.004)$

4. Define $j'(t)$ for all *t* values for which $j'(t)$ exists.

5. Given that $S'(\pi) = \sqrt{7}$ and $S(\pi) = e^2$. Write an equation for the line tangent to the graph of $y = S(m)$ at $m = \pi$.

6. Given $R(x) = \dfrac{2x}{x^2 + 1}$.
   a. Use the limiting value of an appropriate difference quotient to find $R'(x)$.
   b. Write a sentence of two to describe what $y = R'(x)$ defines.
   c. Use your calculator to sketch a graph of both *R* and your R' on the same axes. Write a brief paragraph that describes why your R' graph confirms or refutes your results from part a.

7. Given the graph of $y = f(x)$ on the interval [-2,2] below, provide a reasonable sketch of the graph of f' for the same interval on these axes.



8. Given the graph of $y = g'(x)$ on the interval [-2,2] below, provide a reasonable sketch of the graph of g for the same interval on these axes.