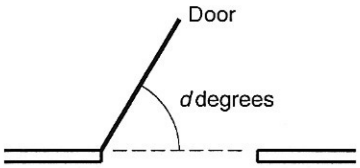
Name: ________________ Group Members: ______________________________

Foerster Calculus Exploration 1-1a: Instantaneous Rate of Change of a Function Date: __________

Objective: Explore the instantaneous rate of change of a function.



The diagram shows a door with an automatic closer. At time $t = 0$ seconds someone pushes the door. It swings open, slows down, stops, starts closing, then slams shut at time $t = 7$ seconds. As the door is in motion the number of degrees, d, it is from its closed position depends on t.

1. Sketch a reasonable graph of d versus t.

2. Suppose that d is given by the equation

$$d = 200t \cdot 2^{-t}.$$

Plot this graph on your grapher. Sketch the results here.

3. Make a table of values of d for each second from $t = 0$ through $t = 10$. Round to the nearest 0.1°.

t	d
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

4. At time $t = 1$ second, does the door appear to be opening or closing? How do you tell?

5. What is the average rate at which the door is moving for the time interval [1, 1.1]? Based on your answer, does the door seem to be opening or closing at time $t = 1$? Explain.

6. By finding average rates for time intervals [1, 1.01], [1, 1.001], and so on, make a conjecture about the *instantaneous* rate at which the door is moving at time $t = 1$ second.

7. Read Section 1-1. What do you notice?!

8. In calculus you will learn by four methods:
   - algebraically,
   - numerically,
   - graphically,
   - verbally (talking and writing).

   Write a paragraph telling what you have learned as a result of doing this Exploration that you did not know before. Use the back of this sheet.

Name ______________________ Group ______________________ Date ______________________

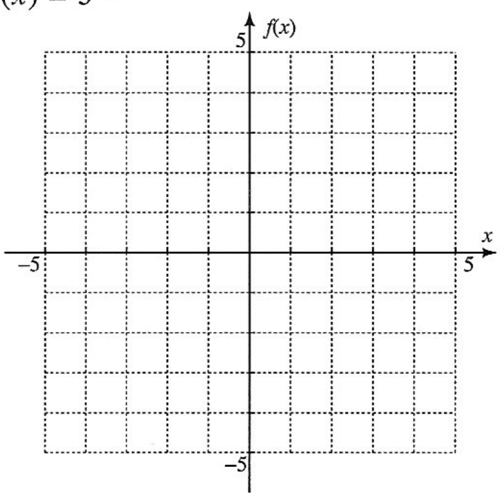
Exploration 1-2a: Graphs of Functions

Objective: Recall the graphs of familiar functions, and tell how fast the function is changing at a particular value of x.

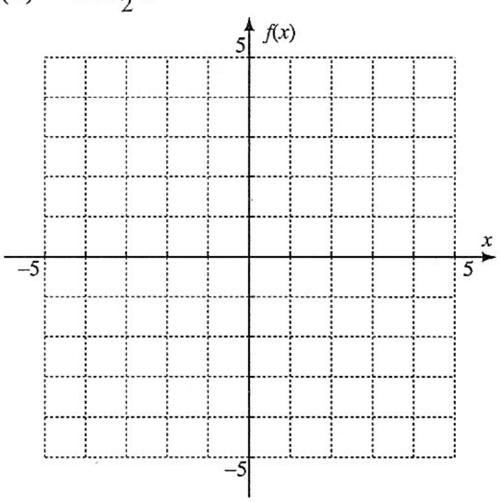
For each function:

a. Without using your grapher, sketch the graph on the axes provided.
b. Confirm by grapher that your sketch is correct.
c. Tell whether the function is increasing, decreasing, or not changing when $x = 1$. If it is increasing or decreasing, tell whether the rate of change is slow or fast.

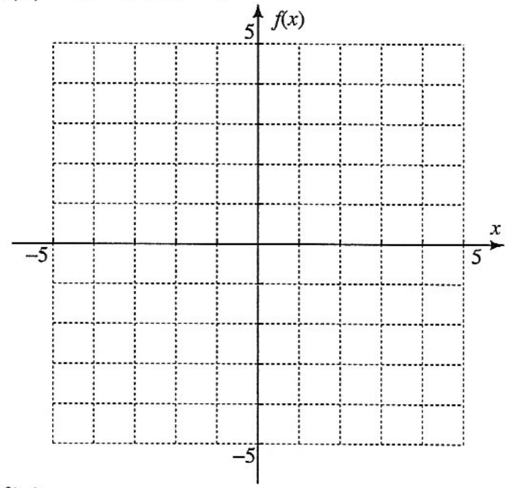
1. $f(x) = 3^{-x}$



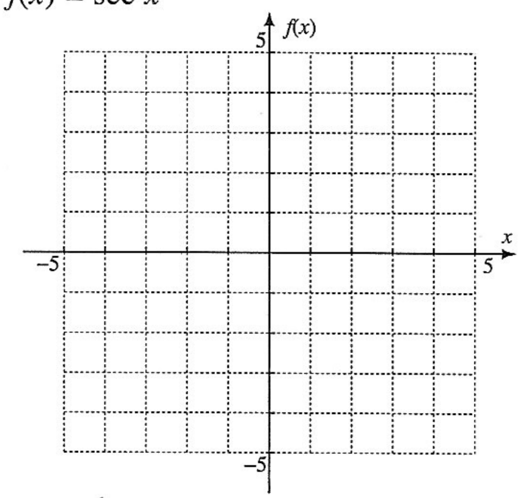
2. $f(x) = \sin \frac{\pi}{2} x$



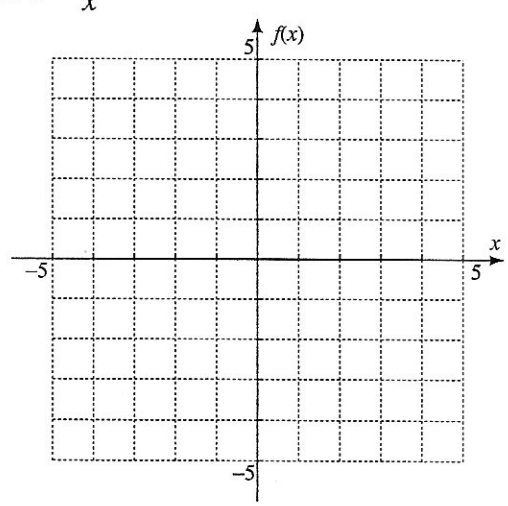
3. $f(x) = x^2 + 2x - 2$



4. $f(x) = \sec x$



5. $f(x) = \dfrac{1}{x}$

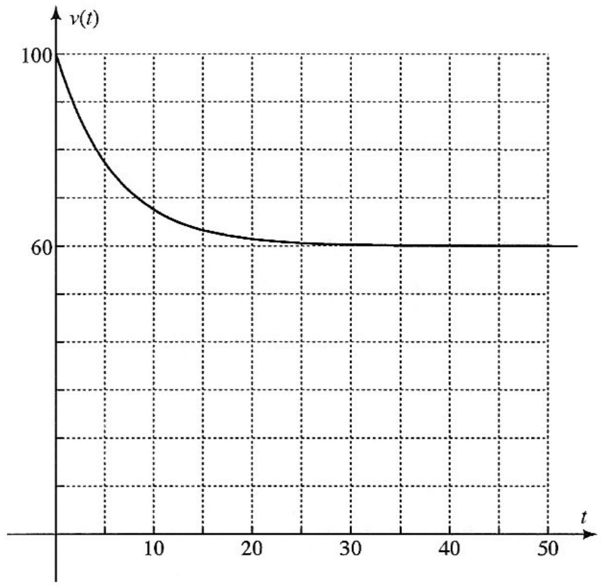


Name: ________________ Group Members: ________________________________

Foerster Calculus Exploration 1-3a: Introduction to Definite Integrals Date: ____________

Objective: Find out what a definite integral is by working a real-world problem concerning speed of a car.

As you drive on the highway you accelerate to 100 feet per second to pass a truck. After you have passed, you slow down to a more moderate 60 ft/sec. The diagram shows the graph of your velocity, $v(t)$, as a function of the number of seconds, t, since you started slowing.



1. What does your velocity seem to be between $t = 30$ and $t = 50$ seconds? How far do you travel in the time interval $[30, 50]$?

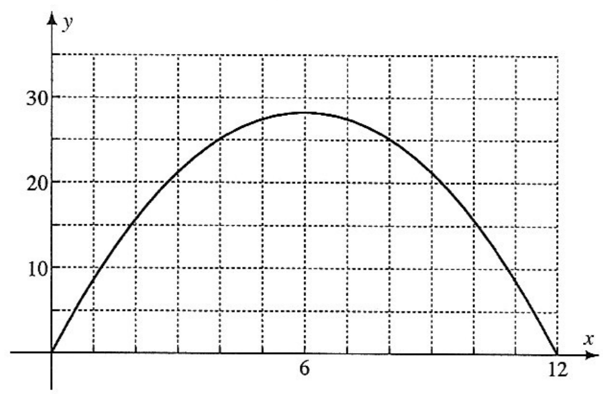
2. Explain why the answer to Problem 1 can be represented as the area of a *rectangular* region of the graph. Shade this region.

3. The distance you travel between $t = 0$ and $t = 20$ can also be represented as the area of a region bounded by the (curved) graph. Count the number of squares in this region. Estimate the area of parts of squares to the nearest 0.1 square space. For instance, how would you count this partial square?

4. How many feet does each small square on the graph represent? How far, therefore, did you go in the time interval $[0, 20]$?

5. Problems 3 and 4 involve finding the product of the x-value and the y-value for a function where y may *vary* with x. Such a product is called the **definite integral** of y with respect to x. Based on the units of t and $v(t)$, explain why the definite integral of $v(t)$ with respect to t in Problem 4 has feet for its units.

6. The graph shows the cross-sectional area, y square inches, of a football as a function of the distance, x inches, from one of its ends. Estimate the definite integral of y with respect to x.



7. What are the units of the definite integral in Problem 6? What, therefore, do you suppose the definite integral represents?

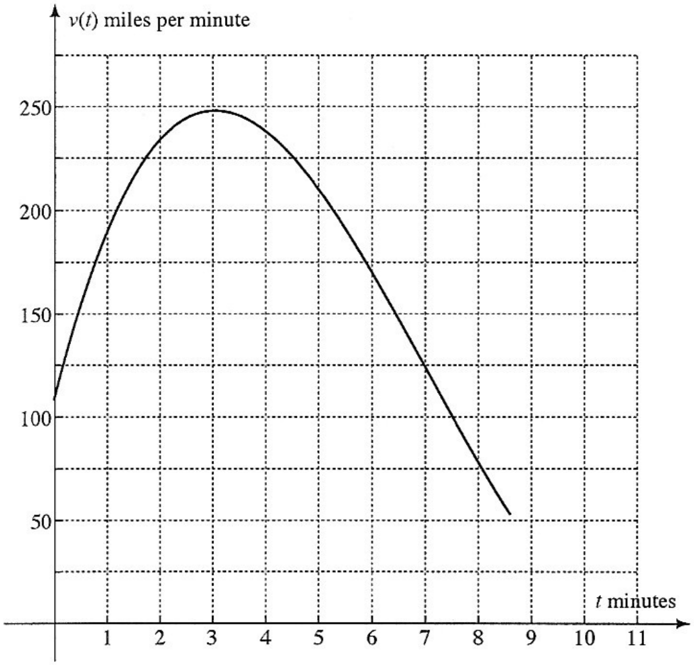
8. What have you learned as a result of doing this Exploration that you did not know before? (Over.)

Name: ________________ Group Members: ____

Foerster Calculus   Exploration 1-4a: Definite Integrals by Trapezoidal Rule   Date: ____________

Objective: Estimate the definite integral of a function numerically instead of graphically by counting squares.

Rocket Problem: Ella Vader (Darth's daughter) is driving in her rocket ship. At time $t = 0$ minutes she fires her rocket engine. The ship speeds up for awhile, then slows down as Alderaan's gravity takes its effect. The graph of her velocity, $v(t)$ miles per minute, is shown below.



1. What mathematical concept would be used to estimate the distance Ella goes between $t = 0$ and $t = 8$?

2. Estimate the distance in Problem 1 geometrically.

3. Ella figures that her velocity is given by

$$v(t) = t^3 - 21t^2 + 100t + 110.$$

Plot this graph on your grapher. Does the graph confirm or refute what Ella figures? Tell how you arrive at your conclusion.

4. Divide the region under the graph from $t = 0$ to $t = 8$, which represents the distance, into four strips of equal width. Draw four trapezoids whose areas approximate the areas of these strips, and whose parallel sides go from the x-axis to the graph. By finding the areas of these trapezoids, estimate the distance Ella goes. Does the answer agree with Problem 2?

5. The technique in Problem 4 is the **trapezoidal rule**. Put a program into your grapher to use this rule. The function equation may be stored as y_1. The input should be the starting time, the ending time, and the number of trapezoids. The output should be the value of the definite integral. Test your program by using it to answer Problem 4.

6. Use the program from Problem 5 to estimate the definite integral using 20 trapezoids.

7. The *exact* value of the definite integral is the *limit* of the estimates by trapezoids as the width of each trapezoid approaches zero. By using the program from Problem 5 make a conjecture about the exact value of the definite integral.

8. What is the fastest Ella went? At what time was that?

9. Approximately what was Ella's rate of change of velocity when $t = 5$? Was she speeding up or slowing down at that time?

10. At what time does Ella stop? Based on the graph, does she stop abruptly or gradually?

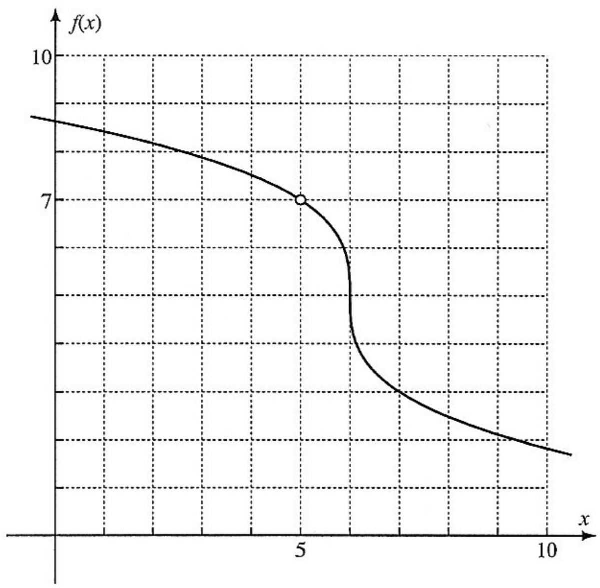
11. What have you learned as a result of doing this Exploration that you did not know before? (Over.)

Name ______________________ Group ______________________ Date ______________

Exploration 2-2a: The Definition of Limit

Objective: Interpret graphically and algebraically the definition of limit.

Let f be the function whose graph is shown here.



1. The limit of $f(x)$ as x approaches 5 is equal to 7. Write the definition as it applies to f at this point.

2. Let $\varepsilon = 1$. From the graph, estimate how close to 5 on the left side x must be kept in order for $f(x)$ to be within ε units of 7.

3. From the graph, estimate how close to 5 on the right side x must be kept in order for $f(x)$ to be within $\varepsilon = 1$ unit of 7.

4. For $\varepsilon = 1$, approximately what is the maximum δ could equal in the definition of limit in order for $f(x)$ to be within ε units of 7 whenever x is within δ units of 5 (but not equal to 5)?

5. The equation of the function graphed is

   $f(x) = 5 - 2(x - 6)^{1/3}$, for $x \neq 5$.

   Calculate precisely the value of δ from Problem 4.

6. If $\varepsilon = 0.01$, calculate precisely the maximum δ could equal in order for $f(x)$ to be within ε units of 7 whenever x is within δ units of 5 (but not equal to 5).

7. Substitute $(7 - \varepsilon)$ for $f(x)$ and $(5 + \delta)$ for x. Solve for δ in terms of ε. Use the result to show that there is a *positive* value of δ for any $\varepsilon > 0$, no matter how small ε is, and thus that 7 really *is* the limit of $f(x)$ as x approaches 5.

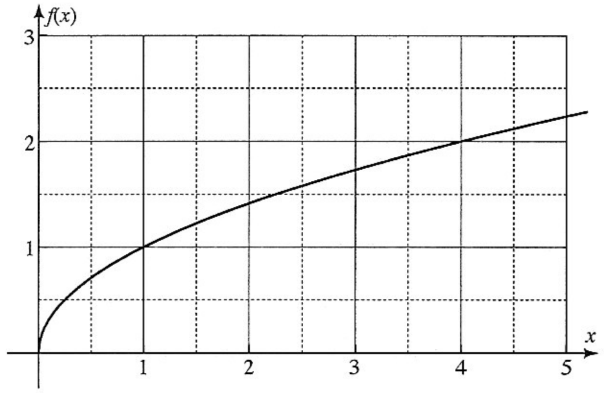
8. What did you learn as a result of doing this Exploration that you did not know before? (Over.)

Name ______________________ Group Members: ____

Exploration 5-4b: Fundamental Theorem of Calculus Preview Date: __________

Objective: Find various Riemann sums for a given definite integral.

The figure shows the graph of $f(x) = \sqrt{x}$.



1. The definite integral of $f(x)$ with respect to x,

   $\int_1^4 \sqrt{x}\, dx$

   is equal to the area of the region under the graph of $f(x)$ between $x = 1$ and $x = 4$. Estimate this integral to one decimal place by counting squares.

2. This integral is also equal to the **limit** of the **Riemann sums**,

   $\lim_{n \to \infty} R_n$

   Estimate this integral again by using your Riemann sums program. Show the sums you use.

3. The **fundamental theorem of calculus**, which you will prove in Section 5-6, says that the *exact* value of a definite integral can be calculated by finding the **indefinite integral**,

   $g(x) = \int f(x)\, dx$

   and evaluating $g(4) - g(1)$. Evaluate this quantity. Show that your answers to Problems 1 and 2 are close to this exact value.

4. Evaluate $\int_3^5 x^3\, dx$ using the fundamental theorem. Ask your instructor for a compact format in which to present your computations.

For Problems 5–8, evaluate the integral using the fundamental theorem.

5. $\int_1^2 \sin x\, dx$

6. $\int_0^{\pi/2} \cos x\, dx$

7. $\int_0^3 e^{2x}\, dx$

8. $\int_{-3}^3 (x^2 + 6x + 21)\, dx$

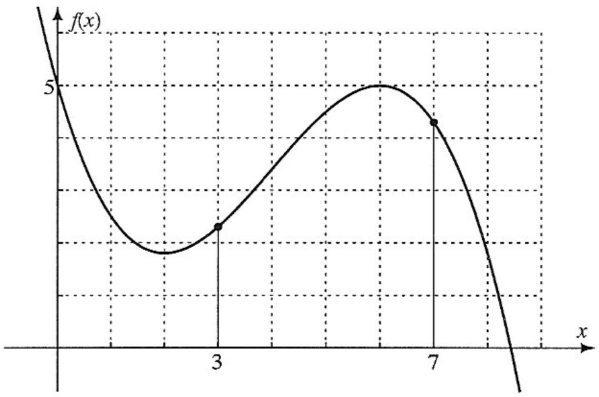
9. What did you learn as a result of doing this Exploration that you did not know before? (Over)

Name ______________________ Group ______________________ Date ______________

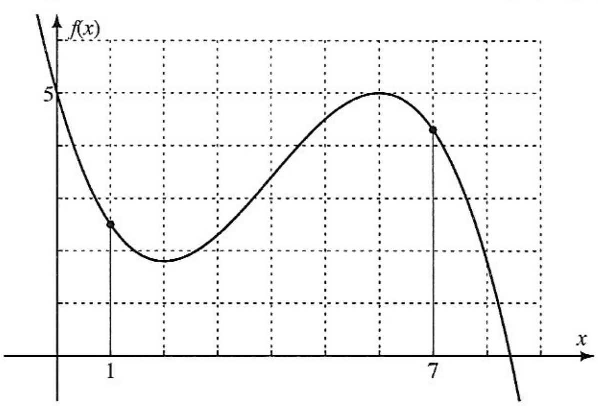
Exploration 5-5a: The Mean Value Theorem

Objective: Without looking at the text, discover the hypotheses and conclusion of the mean value theorem.

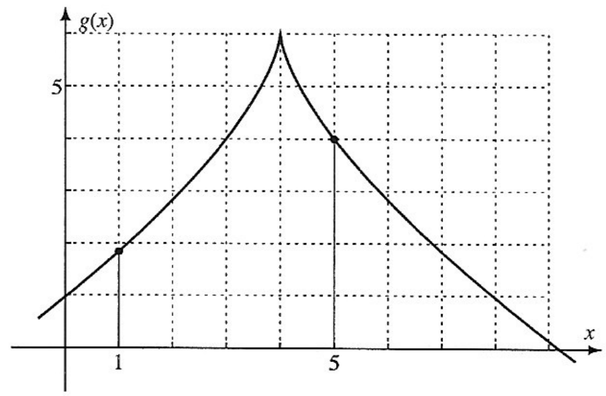
1. For $f(x) = -0.1x^3 + 1.2x^2 - 3.6x + 5$, graphed below, there is a value of $x = c$ between 3 and 7 at which the tangent to the graph is parallel to the secant line through $(3, f(3))$ and $(7, f(7))$.
   - Draw the secant line and the tangent line.
   - From the graph, $c \approx$ ________
   - Is f differentiable on he open interval $(3, 7)$? ____
   - Is f continuous on the closed interval $[3, 7]$? ____



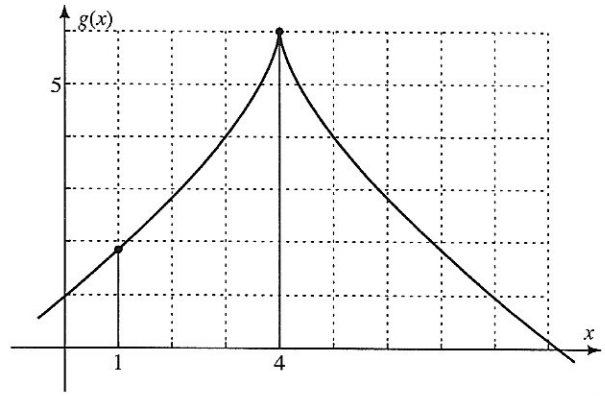
2. Function f from Problem 1 has *two* values of $x = c$ between $x = 1$ and $x = 7$ at which $f'(c)$ equals the slope of the corresponding secant line. (That is, the tangent line parallels the secant line.)
   - Draw the secant and tangents on the graph below.
   - From the graph, $c \approx$ ______ and $c \approx$ _______
   - Is f differentiable on the open interval$(1, 7)$? ____
   - Is f continuous on the closed interval $[1, 7]$? ____



3. For $g(x) = 6 - 2(x - 4)^{2/3}$, graphed below,
   - Draw a secant line through $(1, g(1))$ and $(5, g(5))$
   - Is g differentiable on the open interval $(1, 5)$? ___
   - Is g continuous on the closed interval $[1, 5]$? ___
   - Tell why there is *no* value of $x = c$ between $x = 1$ and $x = 5$ at which $g'(c)$ equals the slope of the secant line.



4. Function g from Problem 3, *does* have a value $x = c$ in $(1, 4)$ for which $g'(c)$ equals the slope of the secant line through $(1, g(1))$ and $(4, g(4))$.
   - Draw the secant line and tangent line, below.
   - From the graph, $c \approx$ ________
   - Is g differentiable on the open interval $(1, 4)$? ___
   - Is g continuous on the closed interval $[1, 4]$? ___

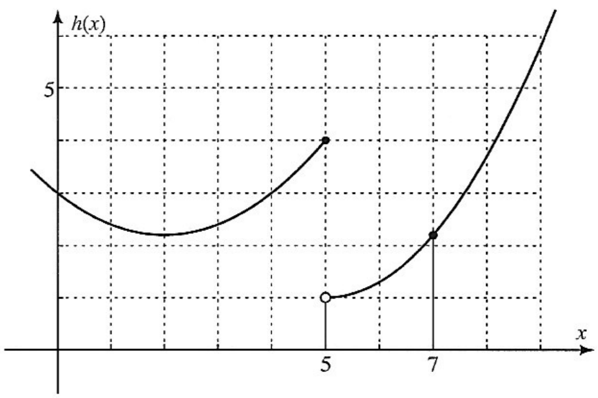


(Other side.)

5. Piecewise function h is defined by

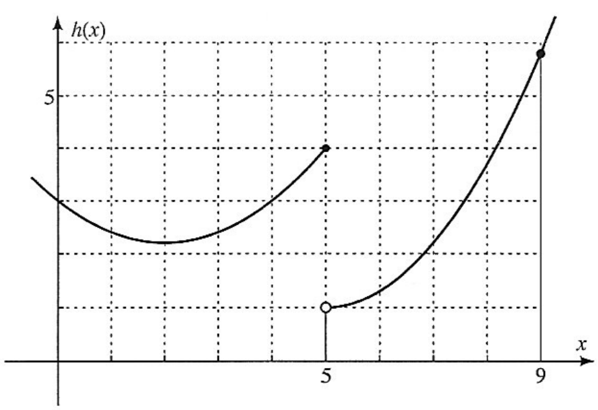
$$h(x) = \begin{cases} 2.2 + 0.2(x - 2)^2 & \text{if } x \le 5 \\ 1 + 0.3(x - 5)^2 & \text{if } x > 5 \end{cases}$$

- Draw a secant line through $(5, h(5))$ and $(7, h(7))$
- Is h differentiable on the open interval $(5, 7)$? ___
- Is h continuous on the closed interval $[5, 7]$? ___
- Why is there *no* value $x = c$ in $(5, 7)$ for which $h'(c)$ equals the slope of the secant line?



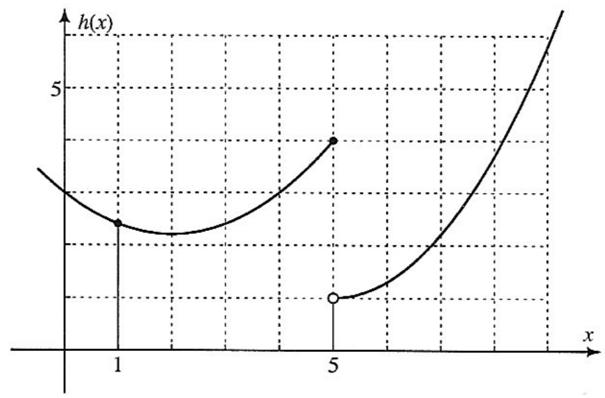
6. The graph below is function h from Problem 5.

- Draw a secant line through $(5, h(5))$ and $(9, h(9))$
- Is h differentiable on the open interval $(5, 9)$? ___
- Is h continuous on the closed interval $[5, 9]$? ___
- There *is* a point $x = c$ in $(5, 9)$ where $h'(c)$ equals the slope of the secant line. Draw the tangent line. Estimate the value of c. _______



7. The graph below is function h from Problem 5

- Draw a secant line through $(1, h(1))$ and $(5, h(5))$
- Show that there is a point $x = c$ in $(1, 5)$ where $h'(c)$ equals the slope of the secant line.
- Is h differentiable on the open interval $(1, 5)$? ___
- Explain why h is continuous on $[1, 5]$, even though there is a step discontinuity at $x = 5$.



8. The mean value theorem states:

If f is differentiable on (a, b) and
f is continuous on $[a, b]$,
then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$ i.e., the secant's slope

For which problem(s) are

- the hypotheses and conclusion true? _________
- the hypotheses and conclusion not true? _______
- the conclusion true, but not the hypotheses? ____

9. The number c is an x-value where the *instantaneous* rate of change equals the *average* ("mean") rate of change. Explain why the hypotheses are **sufficient** conditions for the conclusion, but *not* **necessary** conditions.

10. What did you learn as a result of working this Exploration that you did not know before?

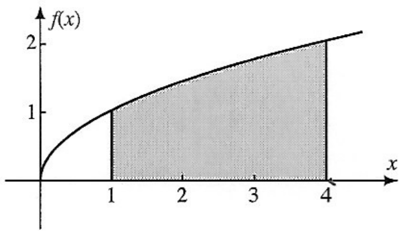
Name ______________________ Group Members: ____

Exploration 5-6a: Some Very Special Riemann Sums

Date: ________

Objective: Calculate Riemann sums for given sets of sample points and reach a conclusion about how the sample points were chosen.

The figure shows the graph of $f(x) = x^{1/2}$. In this Exploration you will integrate $f(x)$ from $x = 1$ to 4.



1. Use the fundamental theorem of calculus to find the exact value of the definite integral

$$I = \int_1^4 x^{1/2}\,dx$$

2. Find an estimate for the integral I by trapezoidal rule with $n = 3$ subintervals. Write down all the decimal places your calculator will give you. Does this value overestimate or underestimate the actual integral?

3. Find a midpoint Riemann sum for integral I in Problem 1. Use $n = 3$ increments. Does the midpoint sum overestimate or underestimate the actual integral? Which is closer to the actual integral, the trapezoidal sum or the Riemann sum?

4. Find a Riemann sum for integral I using the subintervals in Problem 2, but using the following sample points. (k stands for the subinterval number.)

k	$x = c$
1	1.4858425557
2	2.4916102607
3	3.4940272163

How does this sum compare with the exact answer in Problem 1?

5. Find a Riemann sum for I using six subintervals of equal width, and these sample points:

k	$x = c$
1	1.2458051304
2	1.7470136081
3	2.2476804000
4	2.7481034438
5	3.2483958519
6	3.7486100806

How does this sum compare with the exact answer in Problem 1?

6. Let $g(x) = \frac{2}{3}x^{3/2}$. Find the point in the open interval (1, 1.5) at which the conclusion of the mean value theorem is true for function g. Where have you seen this number in this Exploration?

7. Describe verbally how you can find a Riemann sum for a definite integral that is a *constant,* independent of the number of terms in the sum.

8. What did you learn as a result of doing this Exploration that you did not know before? (Over.)

Name: ________________ Group Members: ______________________________________

Foerster Calculus Exploration 5-6b: Proof of the Fundamental Theorem of Calculus Date: __________

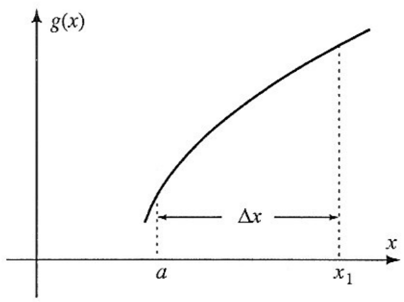
Objective: Prove that a definite integral can be calculated *exactly*, using an indefinite integral.

1. Write the definition of $\int_a^b f(x)\,dx$.

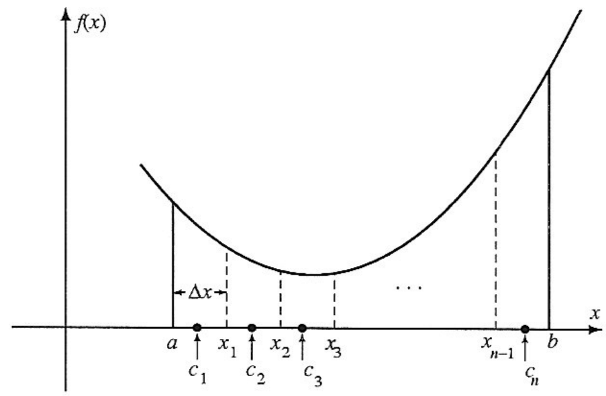
2. Write the definition of $g(x) = \int f(x)\,dx$.

3. How can you be sure that the mean value theorem applies to function g?

4. The figure shows function g in Problem 2. Write the conclusion of the mean value theorem as it applies to g on the interval from $x = a$ to $x = x_1$ and illustrate the conclusion on the graph.



5. The figure in the next column shows the graph of $f(x)$ from Problem 2. Let $c_1, c_2, c_3, \ldots, c_n$ be sample points determined by the mean value theorem as in Problem 4. Write a Riemann sum R_n for $\int_a^b f(x)\,dx$ using these sample points and equal Δx values. Show the Riemann sum on the graph.



6. By the definition of indefinite integral, $f(c_1) = g'(c_1)$. By the mean value theorem, $g'(c_1) = \dfrac{g(x_1) - g(a)}{\Delta x}$, and so on. By appropriate substitutions, show that R_n from Problem 5 is equal to $g(b) - g(a)$.

7. R_n from Problem 6 is *independent* of n, the number of increments. Use this fact, and the fact that $L_n \le R_n \le U_n$ to prove that

$$\int_a^b f(x)\,dx = g(b) - g(a).$$

8. The conclusion in Problem 7 is called the **fundamental theorem of calculus**. Show that you understand what it says by using it to find the *exact* value of $\int_1^4 x^{1/2}\,dx$. (Over.)

Name ______________________ Group Members: ______________________________

Exploration 5-10c Definite Integral as an Accumulated Rate Date: ________

Objective: Apply definite integration to a function whose rate of change is known.

The conclusion of the fundamental theorem is

$$\int_a^b f(x)\,dx = g(b) - g(a)$$

Because the function inside the integral sign is the derivative of the function outside, this conclusion can be written just in terms of a function and its derivative. Replacing $g(x)$ with $f(x)$ and $f(x)$ with $f'(x)$ gives this alternate form of the conclusion.

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Solving for $f(b)$ gives

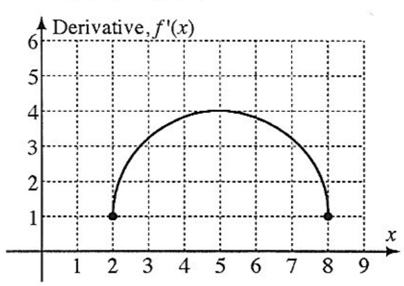
$$f(b) = f(a) + \int_a^b f'(x)\,dx$$

This form of the conclusion says verbally,

> "To find $f(b)$, start at $f(a)$ and add the **accumulated rate of change** of $f(x)$."

1. Demonstrate that you understand this form of the fundamental theorem by finding $f(4)$ if $f(1) = 13$ and $f'(x) = 3x^2$.

2. This form of the fundamental theorem is useful even if you cannot find the integral algebraically. The figure shows the semicircular graph of $f'(x)$. Find $f(8)$ if $f(2) = 7$.



3. Find $f(a)$ if $f(b) = 20$ and $\int_a^b f'(x)\,dx = 9$.

4. **Department Store Problem** – Let $C(x)$ be the number of customers in a particular department store as a function of x hours since opening time. At time $x = 1$ there are 83 customers in the store The table shows $C'(x)$, the rate of change of $C(x)$, (customers/hour in minus customers/hour out) at various times. Estimate the number of customers in the store at time 8 hours. Use an appropriate trapezoidal sum for the integral.

x	Derivative, $C'(x)$
1	200
2	440
5	60
6	–50
8	–160

5. What real-world quanatity does the integral in Problem 4 represent?

6. What did you learn as a result of doing this Exploration that you did not know before?

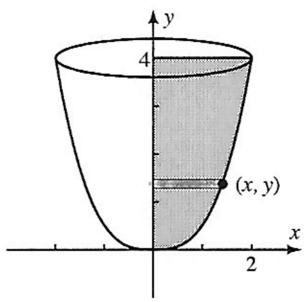
Name ______________________ Group Members: ______________________

Exploration 8-4a: Volumes by Cylindrical Shells

Date: ______

Objective: Find the volume of a solid of revolution by appropriate calculus.

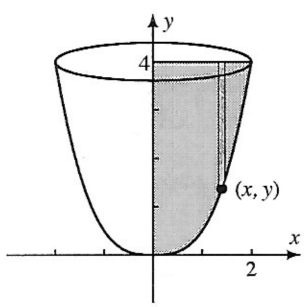
For Problems 1 and 2, the figure shows the region R in Quadrant I bounded by the graph of $y = 0.5x^3$ and the line $y = 4$. The region is rotated about the y-axis to form a solid.

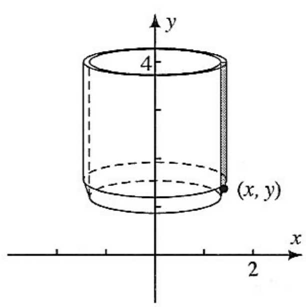


1. The strip shown is perpendicular to the axis of rotation. The sample point (x, y) is on the graph, within the strip. Draw the slice of the figure formed by the strip as it rotates. Then write dV, the volume of the slice, in terms of the sample point.

2. Do the algebra to get dV in terms of one variable. Then find the volume of the entire solid by appropriate integration. You may leave the answer as a multiple of π.

For Problems 3–4, the figure on the left shows the same solid, but with the strip of the region drawn *parallel* to the axis of rotation. As the region rotates, the strip generates a **cylindrical shell**, as shown on the right.

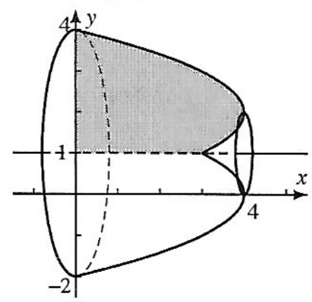




3. Write dV, the volume of the cylindrical shell, in terms of the sample point. Note that the volume equals the circumference of the shell at the sample point times the altitude of the shell at the sample point times the thickness of the shell.

4. Do the algebra to get dV in terms of one variable. Then write an integral equal to the volume of the entire solid. (What are the limits of integration?). Find the volume by doing the integration. If you don't get the same answer as in Problem 2, go back and check your work.

5. The figure shows the solid formed by rotating about the line $y = 1$ the region bounded by the graphs of $x = 4y - y^2$, the y-axis, and the line $y = 1$. Find the volume of the solid by slicing the region parallel to the axis of rotation and using the resulting cylindrical shells.



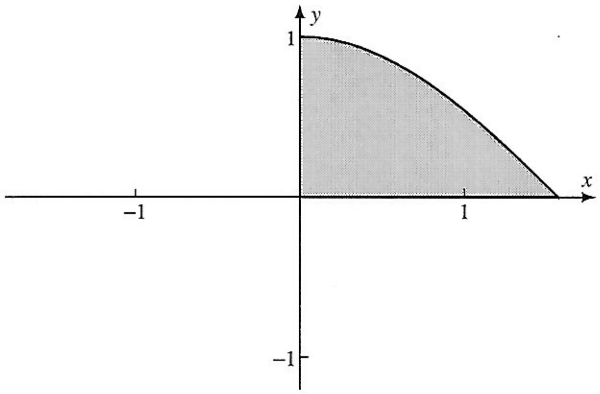
6. What did you learn as a result of doing this exploration that you did not know before? (Over.)

Name ________________________ Group Members: ____

Exploration 5-9e: Volume Game! Date: __________

Objective: Find the volume of various solids by plane slicing.

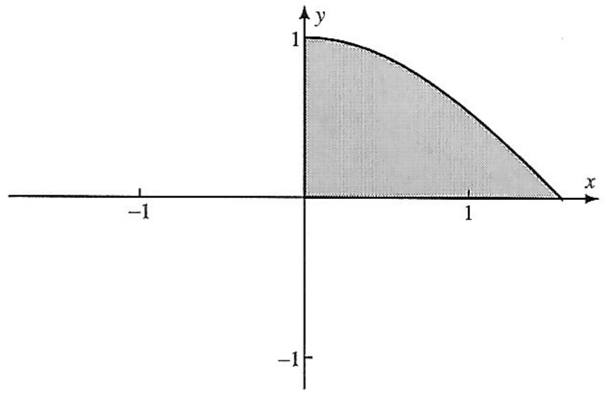
The figure shows the region under a quarter-cycle of the graph of $y = \cos x$.



1. Draw a sample point on the graph and a representative slice of the region containing this sample point. Use the results to find the area of the region.

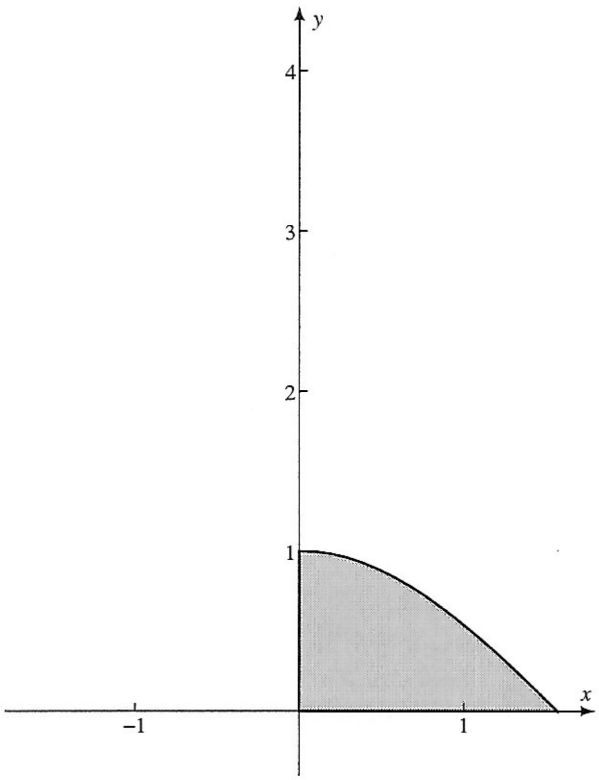
2. On the given figure, sketch the solid formed by rotating the region about the x-axis. Show the disk formed as the slice of the region rotates. Find the volume of the solid. Is the answer an "interesting" multiple of π?

3. On this copy of the figure, sketch the graph of the solid formed by rotating the region about the y-axis. Show a representative slice of the region, the corresponding sample point, and the disk formed as this slice rotates. Find the volume of the solid.

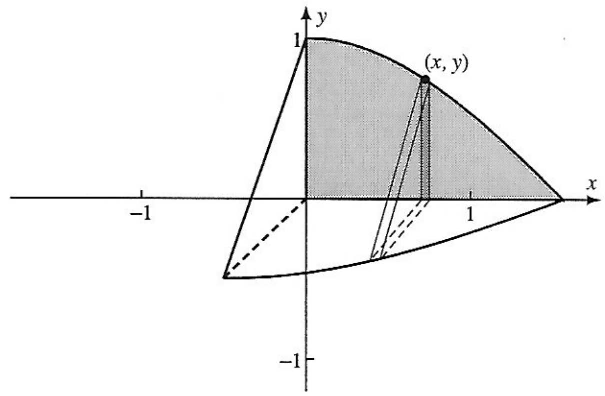


(Other side)

4. On this copy of the figure from the other side, sketch the solid formed by rotating the region about the line $y = 2$. Show a representative slice of the region, the corresponding sample point, and the washer formed as this slice rotates. Find the volume of the solid.



5. The figure shows the region from the other problems in this Exploration. The region forms the base of a solid. Each cross-section perpendicular to the x-axis is an isosceles right triangle with one leg in the base of the solid. Find dV, the volume of the triangular slice of this solid shown in the figure. Then integrate to find the volume of the solid.



6. What did you learn as a result of doing this Exploration that you did not know before?

Name ______________________ Group Members: ____

Exploration 11-1a: Chair Work Problem

Date: ____________

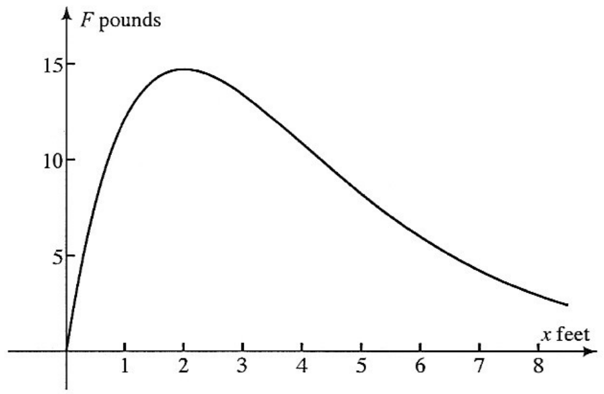
Objective: Find the work done in dragging a chair across the floor.

In physics you learn that work, W, done in moving an object from one place to another is defined to be the force exerted on the object times the distance the object moves. In this Exploration you will calculate the work done by a *variable* force.

Suppose that you push a chair across the floor with a force F in pounds given by the equation

$$F = 20xe^{-0.5x}$$

where x is the distance in feet that the chair has moved from its starting point. The figure shows the graph of F as a function of x.



1. Draw a narrow vertical strip of width dx centered at a sample point on the graph where $x = 4$. Approximately what is the force at any value of x on the graph in this strip? Approximately how much work is done in moving the chair a distance dx feet in this strip?

2. Write an equation for dW, the work done in moving the chair a distance dx when x is at 4 feet.

3. Which of the four concepts of calculus would you use to add up all the values of dW as the chair moves from $x = 0$ to $x = 7$, and find the limit of this sum as dx approaches zero and thus the number of strips approaches infinity?

4. Find the work done in moving the chair from $x = 0$ to $x = 7$. Why is "foot-pounds" an appropriate name for the units of work?

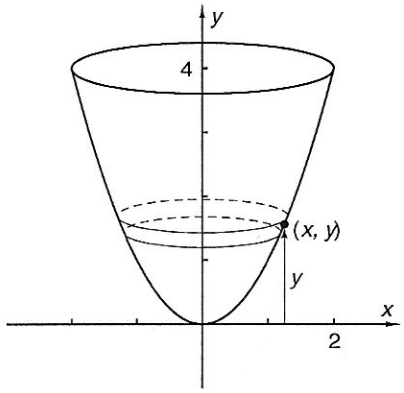
5. If you continue to push the chair with a force F given by the equation, and the chair continues to move, what limit would the amount of work approach at x approaches infinity? Name the calculus topic you use to answer this question.

8. What did you learn as a result of doing this exploration that you did not know before?

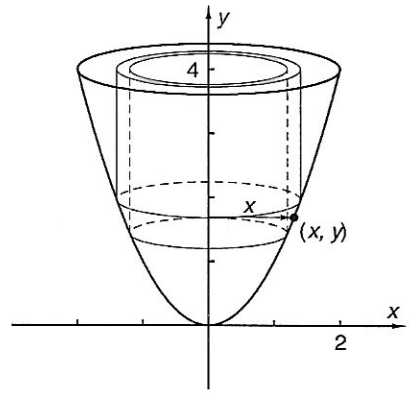
Name: ____________________ Group Members: ______________________________

Foerster Calculus Exploration 11-3: Mass of a Variable-Density Solid Date: ______________

Objective: Find the mass of a solid of revolution if its density varies axially or radially.



1. The figure shows the paraboloid formed by rotating about the y-axis the region above the graph of $y = x^2$, below $y = 4$, and to the right of the y-axis, where x and y are in centimeters. Assume that the density of the solid varies **axially** (in the direction of the axis of the solid), being equal to $3y^{1/2}$ grams per cubic centimeter at a sample point (x, y) in a horizontal disk. Find the mass, *dm,* of the disk in terms of the sample point. Then use appropriate calculus to find the mass of the entire solid.



2. The figure shows another solid congruent to the solid in Problem 1. The density of this solid varies **radially** (in the direction of the radius), being equal to $x + 5$ grams per cubic centimeter at a sample point (x, y) in a cylindrical shell. Find the mass, *dm,* of the cylindrical shell in terms of the sample point. Then use appropriate calculus to find the mass of the entire solid.

3. What did you learn as a result of doing this Exploration that you did not know before?