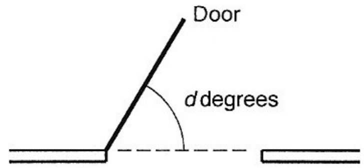
Name: ________________ Group Members: ________________________________

Foerster Calculus Solutions, Exploration 1-1a: Instantaneous Rate of Change of a Function Date: ________

Objective: Explore the instantaneous rate of change of a function.



The diagram shows a door with an automatic closer. At time $t = 0$ seconds someone pushes the door. It swings open, slows down, stops, starts closing, then slams shut at time $t = 7$ seconds. As the door is in motion the number of degrees, d, it is from its closed position depends on t.

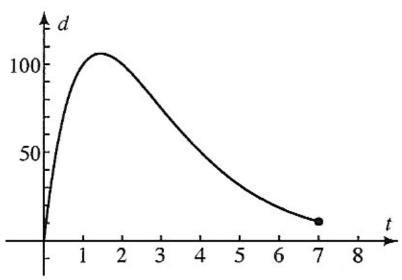
1. Sketch a reasonable graph of d versus t.

   • Any graph is reasonable that starts at the origin, rises, reaches a maximum, drops toward zero degrees, and stops at $t = 7$ seconds.

2. Suppose that d is given by the equation

   $d = 200t \cdot 2^{-t}$.

   Plot this graph on your grapher. Sketch the results here.



3. Make a table of values of d for each second from $t = 0$ through $t = 10$. Round to the nearest 0.1°.

t	d
0	0
1	100
2	100
3	75
4	50
5	31.25
6	18.75
7	10.93...
8	6.25
9	3.51...
10	1.95...

4. At time $t = 1$ second, does the door appear to be opening or closing? How do you tell?

   • Opening.

   • The graph is still increasing at $t = 1$.

5. What is the average rate at which the door is moving for the time interval [1, 1.1]? Based on your answer, does the door seem to be opening or closing at time $t = 1$? Explain.

   • $d() = 100$ and $d(1.1) = 102.63362...$

   • Door opened by 2.63362... degrees in 0.1 s.

   • Av. rate $= \dfrac{2.63362...}{0.1} = 26.3362... \approx 26.34°/\text{s}$

6. By finding average rates for time intervals [1, 1.01], [1, 1.001], and so on, make a conjecture about the *instantaneous* rate at which the door is moving at time $t = 1$ second.

   • [1, 1.01]: 30.2342... °/s

   • [1, 1.001]: 30.6400... °/s

   • [1, 1.0001]: 30.6807... °/s

   • Conjecture: Any number slightly *above* 30.6807... is a reasonable conjecture. Some students will conjecture 31°/s, thinking, "All answers in mathematics are whole numbers." The exact answer is 30.6853...°/s.

7. Read Section 1-1. What do you notice?!

   • This problem is the Example in Section 1-1.

8. In calculus you will learn by four methods:
   • algebraically,
   • numerically,
   • graphically,
   • verbally (talking and writing).
   Write a paragraph telling what you have learned as a result of doing this Exploration that you did not know before. Use the back of this sheet.

   • Answers will vary.

Name ______________________________ Group ______________________________ Date ______________________

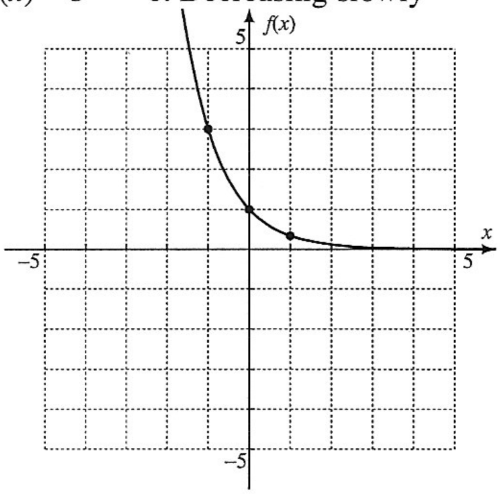
Solutions, Exploration 1-2a: Graphs of Functions

Objective: Recall the graphs of familiar functions, and tell how fast the function is changing at a particular value of *x*.

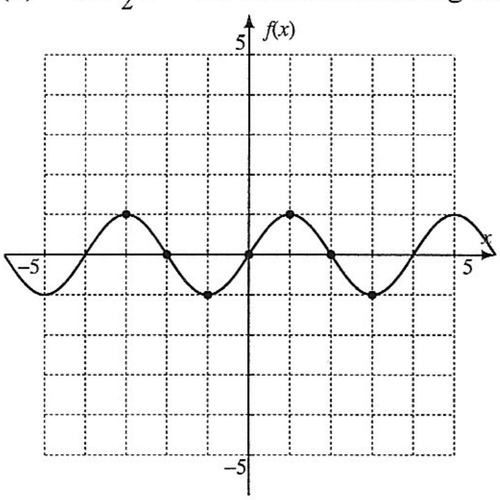
For each function:

a. Without using your grapher, sketch the graph on the axes provided.
b. Confirm by grapher that your sketch is correct.
c. Tell whether the function is increasing, decreasing, or not changing when $x = 1$. If it is increasing or decreasing, tell whether the rate of change is slow or fast.

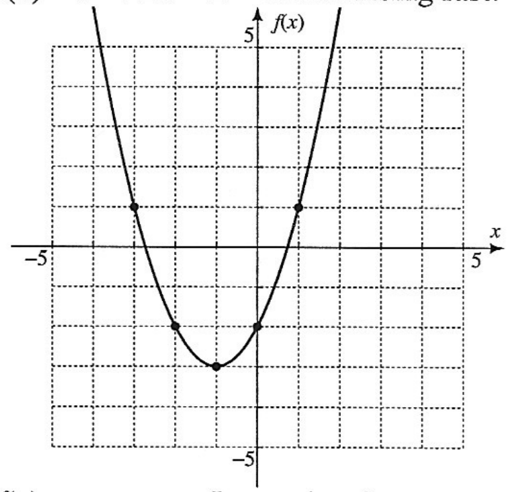
1. $f(x) = 3^{-x}$ • c. Decreasing slowly



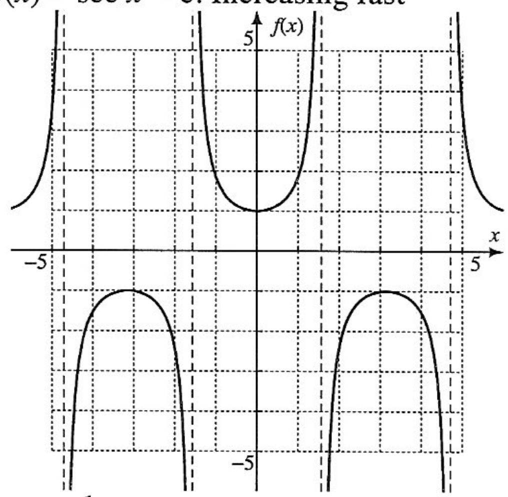
2. $f(x) = \sin \frac{\pi}{2} x$ • c. Neither increasing nor decreasing.



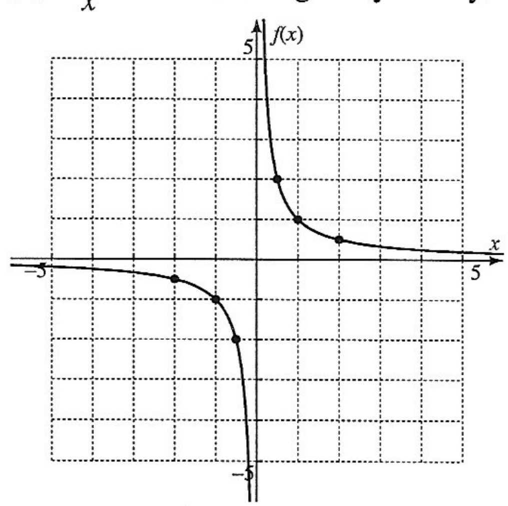
3. $f(x) = x^2 + 2x - 2$ • c. Increasing fast.



4. $f(x) = \sec x$ • c. Increasing fast



5. $f(x) = \dfrac{1}{x}$ • c. Decreasing fairly slowly.

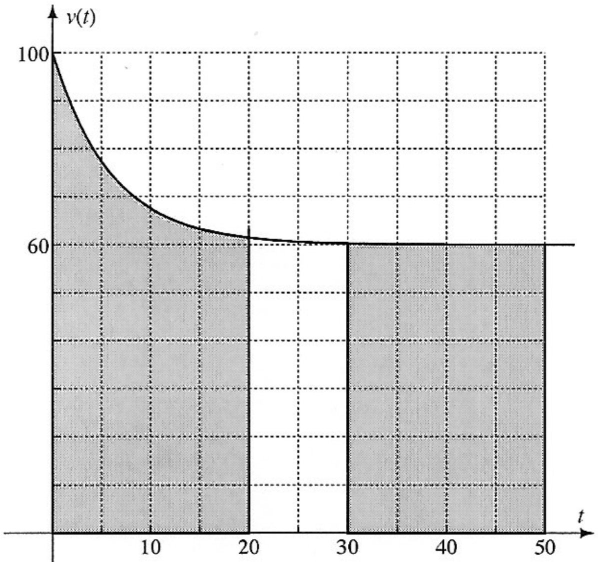


Name: ________________ Group Members: ______________________________________

Foerster Calculus   Solutions, Exploration 1-3a: Introduction to Definite Integrals   Date: ____________

Objective: Find out what a definite integral is by working a real-world problem concerning speed of a car.

As you drive on the highway you accelerate to 100 feet per second to pass a truck. After you have passed, you slow down to a more moderate 60 ft/sec. The diagram shows the graph of your velocity, $v(t)$, as a function of the number of seconds, t, since you started slowing.



1. What does your velocity seem to be between $t = 30$ and $t = 50$ seconds? How far do you travel in the time interval $[30, 50]$?

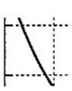
   • From $t = 30$ to $t = 50$ s, the velocity seems to be about 60 ft/s. Distance = rate · time, so the distance traveled is about 60 ft/s·(50 – 30)s = 1200 ft.

2. Explain why the answer to Problem 1 can be represented as the area of a *rectangular* region of the graph. Shade this region.

   • The rectangle on the graph has height = 60 and base from 30 to 50 has area base · height = 1200.

   • See graph above Problem 1.

3. The distance you travel between $t = 0$ and $t = 20$ can also be represented as the area of a region bounded by the (curved) graph. Count the number of squares in this region. Estimate the area of parts of squares to the nearest 0.1 square space. For instance, how would you count this partial square?

    • About 0.7 square.

   • All the squares and partial squares under the graph from $t = 0$ to $t = 20$ have area about 28.6 square spaces.

4. How many feet does each small square on the graph represent? How far, therefore, did you go in the time interval $[0, 20]$?

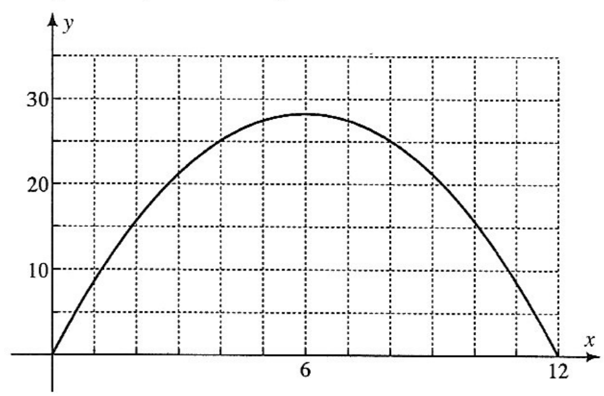
   • Each small square has base representing 5 s and height representing 10 ft/s. So the area of each small square = base · height represents 50 ft. Therefore the distance was about 28.6 · 50 = 1430 ft. (Exact ans. is 1431.3207....)

5. Problems 3 and 4 involve finding the product of the x-value and the y-value for a function where y may *vary* with x. Such a product is called the **definite integral** of y with respect to x. Based on the units of t and $v(t)$, explain why the definite integral of $v(t)$ with respect to t in Problem 4 has units "feet."

   • The x-value is in seconds and the y-value is in feet/second. So their product (i.e., the definite integral) is in

   • seconds $\cdot \dfrac{\text{feet}}{\text{second}} =$ feet

6. The graph shows the cross-sectional area, y square inches, of a football as a function of the distance, x inches, from one of its ends. Estimate the definite integral of y with respect to x.



   • About 45.2 square spaces, each with base 1 and height 5. So each square space represents 5 units of the definite integral. 5 · 45.2 = 226 square units. (Exactly 226.1946...)

7. What are the units of the definite integral in Problem 6? What, therefore, do you suppose the definite integral represents?

   • (x-units)·(y-units) = (in.)·(in.^{2}) = in.^{3}, the *volume* of the football.

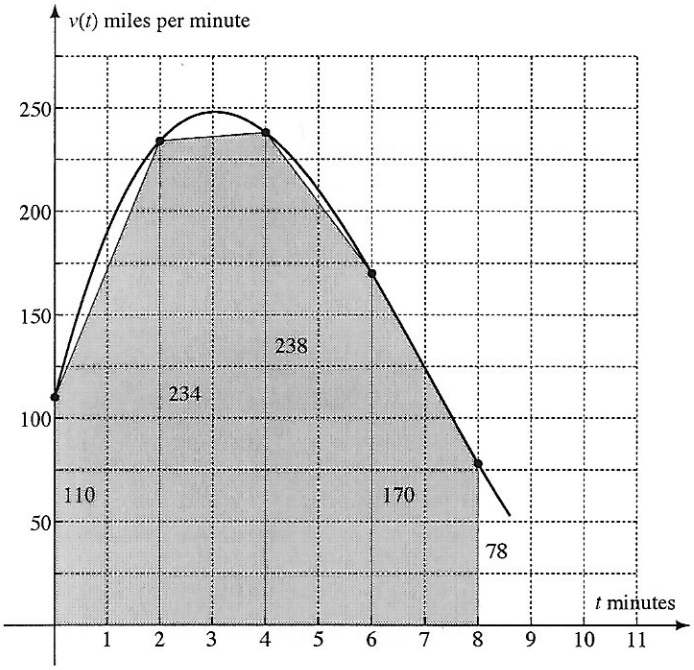
8. What have you learned as a result of doing this Exploration that you did not know before?
   • Answers will vary.

Name: ________________ Group Members: ______________________________

Foerster Calculus Solutions, Exploration 1-4a: Definite Integrals by Trapezoidal Rule Date: __________

Objective: Estimate the definite integral of a function numerically instead of graphically by counting squares.

Rocket Problem: Ella Vader (Darth's daughter) is driving in her rocket ship. At time $t = 0$ minutes she fires her rocket engine. The ship speeds up for awhile, then slows down as Alderaan's gravity takes its effect. The graph of her velocity, $v(t)$ miles per minute, is shown below.



1. What mathematical concept would be used to estimate the distance Ella goes between $t = 0$ and $t = 8$?

   • Definite integral. (time)(velocity) = distance.

2. Estimate the distance in Problem 1 geometrically.

   • There are about 60.8 squares under the graph, each representing (1)(25) = 25 miles. So the total distance is about (60.8)(25) = 1520 miles.

3. Ella figures that her velocity is given by

   $v(t) = t^3 - 21t^2 + 100t + 110.$

   Plot this graph on your grapher. Does the graph confirm or refute what Ella figures? Tell how you arrive at your conclusion.

   • By trace or by table, the values of $v(t)$ for integer values of x confirm the values shown on the graph.

4. Divide the region under the graph from $t = 0$ to $t = 8$, which represents the distance, into four strips of equal width. Draw four trapezoids whose areas approximate the areas of these strips, and whose parallel sides go from the x-axis to the graph. By finding the areas of these trapezoids, estimate the distance Ella goes. Does the answer agree with Problem 2? • Graph, above Problem 1.

   • Area = 344 + 472 + 408 + 248 = 1472. • Close.

5. The technique in Problem 4 is the **trapezoidal rule**. Put a program into your grapher to use this rule. The function equation may be stored as y_1. The input should be the starting time, the ending time, and the number of trapezoids. The output should be the value of the definite integral. Test your program by using it to answer Problem 4.
   • Answer for $n = 4$ is 1472, which agrees.

6. Use the program from Problem 5 to estimate the definite integral using 20 trapezoids.

   • 20 trapezoids: integral ≈ 1518.08 mi.

7. The *exact* value of the definite integral is the *limit* of the estimates by trapezoids as the width of each trapezoid approaches zero. By using the program from Problem 5 make a conjecture about the exact value of the definite integral.
   • 15519.52 miles for 40 trapezoids
   • 1519.999232 miles for 1000 trapezoids
   • Conjecture: Exact value is 1520 miles.

8. What is the fastest Ella went? At what time was that?

   • By graph, about 248 mi/min at 3 minutes (Exactly 248.0209... at $7 - \sqrt{47/3}$ minutes)

9. Approximately what was Ella's rate of change of velocity when $t = 5$? Was she speeding up or slowing down at that time?

   • $\dfrac{v(5.1) - v(4.9)}{5.1 - 4.9} = -34.99$
   • Slowing at about 35 (mi/min)/min

10. At what time does Ella stop? Based on the graph, does she stop abruptly or gradually?

    • Ella stops at 11 minutes because $v(11) = 0$.
    • The stop is gradual because the graph just touches the horizontal axis, tangent to it.

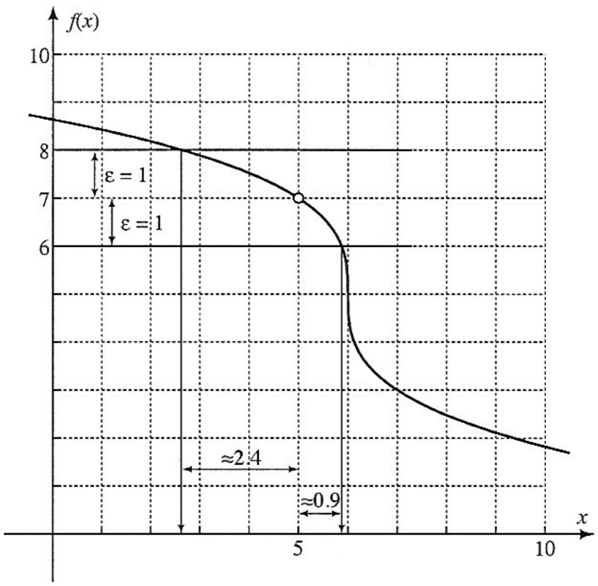
11. What have you learned as a result of doing this Exploration that you did not know before?
    • Answers will vary.

Name ______________________________ Group ________________________ Date ________________

Solutions, Exploration 2-2a: The Definition of Limit

Objective: Interpret graphically and algebraically the definition of limit.

Let f be the function whose graph is shown here.



1. The limit of $f(x)$ as x approaches 5 is equal to 7. Write the definition as it applies to f at this point.

   • 7 is the limit of $f(x)$ as x approaches 5 if and only if
   • for any positive number ε, no matter how small,
   • there is a positive number δ such that
   • if x is kept within δ units of 5, but $x \neq 5$,
   • then $f(x)$ stays within ε units of 7.

2. Let $\varepsilon = 1$. From the graph, estimate how close to 5 on the left side x must be kept in order for $f(x)$ to be within ε units of 7.

   • Graph, above Problem 1.

   • Left side: Within about 2.4 units of 5.

3. From the graph, estimate how close to 5 on the right side x must be kept in order for $f(x)$ to be within $\varepsilon = 1$ unit of 7.

   • Graph, above Problem 1.

   • Within about 0.9 units of 5.

4. For $\varepsilon = 1$, approximately what is the maximum δ could equal in the definition of limit in order for $f(x)$ to be within ε units of 7 whenever x is within δ units of 5 (but not equal to 5)?

   • Maximum $\delta \approx 0.9$ units (Think!!)

5. The equation of the function graphed is

   $f(x) = 5 - 2(x-6)^{1/3}$, for $x \neq 5$.

   Calculate precisely the value of δ from Problem 4.

   • $6 = 5 - 2(x-6)^{1/3}$ Substitute 6 for $f(x)$.

   • $(x-6)^{1/3} = -0.5$

   • $x - 6 = -0.125$

   • $x = 5.875$ Solve for x.

   • $\delta = 5.875 - 5 = \underline{0.875}$, confirming $\delta \approx 0.9$ $x = 5 + \delta$

6. If $\varepsilon = 0.01$, calculate precisely the maximum δ could equal in order for $f(x)$ to be within ε units of 7 whenever x is within δ units of 5 (but not equal to 5).

   • $6.99 = 5 - 2(x-6)^{1/3}$ Substitute 6.99 for $f(x)$.

   • $(x-6)^{1/3} = -0.995$

   • $x - 6 = -0.985074...$

   • $x = 5.014925...$ Solve for x.

   • $\delta = 5.014925... - 5 = \underline{0.014925...}$ $x = 5 + \delta$

7. Substitute $(7 - \varepsilon)$ for $f(x)$ and $(5 + \delta)$ for x. Solve for δ in terms of ε. Use the result to show that there is a *positive* value of δ for any $\varepsilon > 0$, no matter how small ε is, and thus that 7 really *is* the limit of $f(x)$ as x approaches 5.

   • $(7 - \varepsilon) = 5 - 2((5+\delta) - 6)^{1/3}$ $x = 5 + \delta$

   • $2((5+\delta) - 6)^{1/3} = -2 + \varepsilon$

   • $(\delta - 1)^{1/3} = -1 + 0.5\varepsilon$

   • $\delta - 1 = (-1 + 0.5\varepsilon)^3$

   • $\delta = 1 + (-1 + 0.5\varepsilon)^3$

   • Because $-1 + 0.5\varepsilon$ is greater than -1 for positive values of ε, the quantity $1 + (-1 + 0.5\varepsilon)^3$ is greater than 0 for positive values of ε, no matter how small.

   • That is, $\delta > 0$ for any $\varepsilon > 0$, no matter how small.

   • $\therefore$ 7 really is the limit of $f(x)$ as x approaches 7, by the definition of limit, Q.E.D.

8. What did you learn as a result of doing this Exploration that you did not know before?

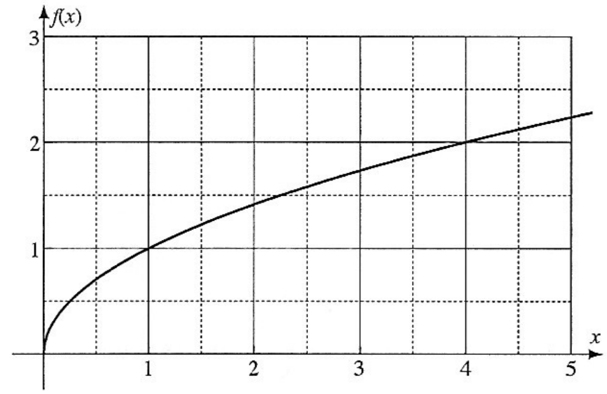
   • Answers will vary.

Name ______________________ Group Members: ______________________

Solutions, Exploration 5-4b: Fundamental Theorem of Calculus Preview

Objective: Find various Riemann sums for a given definite integral.

The figure shows the graph of $f(x) = \sqrt{x}$.



1. The definite integral of $f(x)$ with respect to x,

   $$\int_1^4 \sqrt{x}\, dx$$

   is equal to the area of the region under the graph of $f(x)$ between $x = 1$ and $x = 4$. Estimate this integral to one decimal place by counting squares.

   - $\int_1^4 \sqrt{x}\, dx \approx$ <u>4.6</u> or <u>4.7</u>

2. This integral is also equal to the **limit** of the **Riemann sums**,

   $$\lim_{n\to\infty} R_n$$

   Estimate this integral again by using your Riemann sums program. Show the sums you use.

   - Using midpoint Riemann sums,
   - $M_{10} = 4.66760066...$
     $M_{100} = 4.66667604...$
     $M_{500} = 4.66666704...$

3. The **fundamental theorem of calculus**, which you will prove in Section 5-6, says that the *exact* value of a definite integral can be calculated by finding the **indefinite integral**,

   $$g(x) = \int f(x)\, dx$$

   and evaluating $g(4) - g(1)$. Evaluate this quantity. Show that your answers to Problems 1 and 2 are close to this exact value.

   - $g(x) = \int x^{1/2}\, dx = \frac{2}{3}x^{3/2} + C$
   - $g(4) - g(1) = (\frac{2}{3} \cdot 4^{3/2} + C) - (\frac{2}{3} \cdot 1^{3/2} + C)$
   - $= 4\frac{2}{3} = 4.66666666...$, close!

4. Evaluate $\int_3^5 x^3\, dx$ using the fundamental theorem. Ask your instructor for a compact format in which to present your computations.

   - $\int_3^5 x^3\, dx = \frac{1}{4}x^4 \Big|_3^5$
   - $= \frac{1}{4}5^4 - \frac{1}{4}3^4$
   - $=$ <u>136</u>

For Problems 5–8, evaluate the integral using the fundamental theorem.

5. $\int_1^2 \sin x\, dx$

   - $= -\cos x \Big|_1^8$
   - $= -\cos 2 + \cos 1$
   - $=$ <u>−0.9564...</u>

6. $\int_0^{\pi/2} \cos x\, dx$

   - $= \sin x \Big|_0^{\pi/2}$
   - $= \sin\frac{\pi}{2} - \sin =$
   - $=$ <u>1</u> (remarkable!)

7. $\int_0^3 e^{2x}\, dx$

   - $= \frac{1}{2}e^{2x} \Big|_0^3$
   - $= \frac{1}{2}e^6 - \frac{1}{2}e^0$
   - $=$ <u>201.2143...</u>

8. $\int_{-3}^3 (x^2 + 6x + 21)\, dx$

   - $= \frac{1}{3}x^3 + 3x^2 + 21x \Big|_{-3}^3$
   - $= 9 + 27 + 63 - (-9 + 27 - 63)$
   - $=$ <u>144</u>

9. What did you learn as a result of doing this Exploration that you did not know before?

   - Answers will vary, but should include the fundamental theorem of calculus.

Name ______________________ Group ______________________ Date ______________

Solutions, Exploration 5-5a: The Mean Value Theorem

Objective: Without looking at the text, discover the hypotheses and conclusion of the mean value theorem.

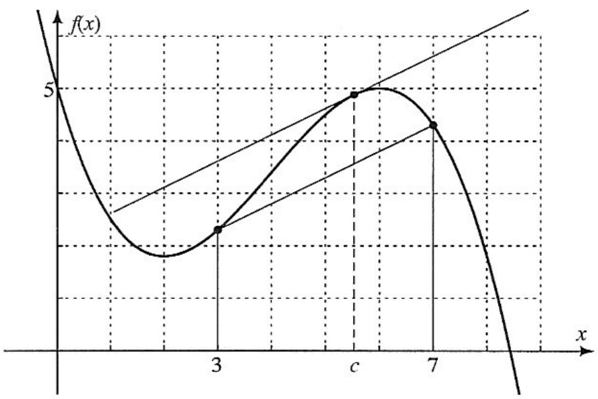
1. For $f(x) = -0.1x^3 + 1.2x^2 - 3.6x + 5$, graphed below, there is a value of $x = c$ between 3 and 7 at which the tangent to the graph is parallel to the secant line through $(3, f(3))$ and $(7, f(7))$.

   Draw the secant line and the tangent line. • graph

   From the graph, $c \approx$ 5.5

   Is f differentiable on he open interval $(3, 7)$? • Yes

   Is f continuous on the closed interval $[3, 7]$? • Yes



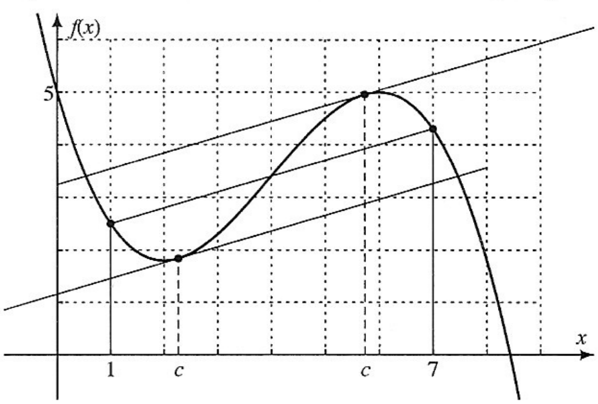
2. Function f from Problem 1 has *two* values of $x = c$ between $x = 1$ and $x = 7$ at which $f'(c)$ equals the slope of the corresponding secant line. (That is, the tangent line parallels the secant line.)

   Draw the secant and tangents on the graph below.

   From the graph, $c \approx$ • 2.3 and $c \approx$ • 5.7

   Is f differentiable on the open interval$(1, 7)$? • Yes

   Is f continuous on the closed interval $[1, 7]$? • Yes



3. For $g(x) = 6 - 2(x - 4)^{2/3}$, graphed below,

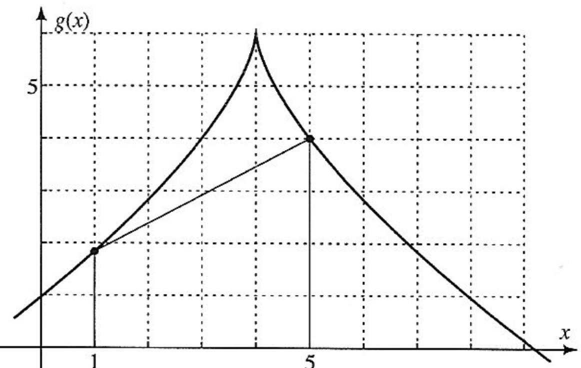
   Draw a secant line through $(1, g(1))$ and $(5, g(5))$

   Is g differentiable on the open interval $(1, 5)$? • No

   Is g continuous on the closed interval $[1, 5]$? • Yes

   Tell why there is *no* value of $x = c$ between $x = 1$ and $x = 5$ at which $g'(c)$ equals the slope of the secant line.

   • The slope of the secant line is positive. For x between 1 and 4 the tangent lines are too steep. For x between 4 and 5 the tangent lines have negative slope. (At $x = 4$ the tangent line is vertical, and thus has no slope.)



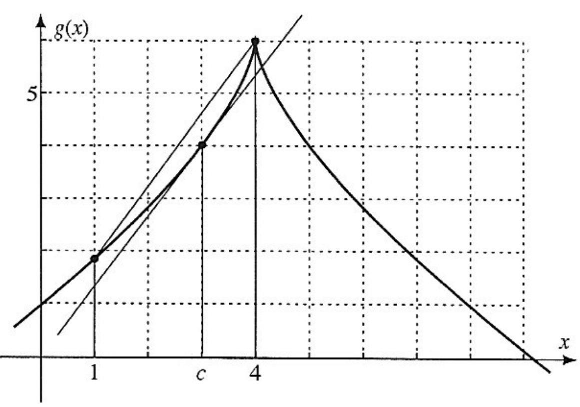
4. Function g from Problem 3, *does* have a value $x = c$ in $(1, 4)$ for which $g'(c)$ equals the slope of the secant line through $(1, g(1))$ and $(4, g(4))$.

   Draw the secant line and tangent line, below.

   From the graph, $c \approx$ • 3.0

   Is g differentiable on the open interval $(1, 4)$? • yes

   Is g continuous on the closed interval $[1, 4]$? • no



(Other side.)

5. Piecewise function h is defined by

$$h(x) = \begin{cases} 2.2 + 0.2(x-2)^2 & \text{if } x \le 5 \\ 1 + 0.3(x-5)^2 & \text{if } x > 5 \end{cases}$$

Draw a secant line through $(5, h(5))$ and $(7, h(7))$

Is h differentiable on the open interval $(5, 7)$? • yes

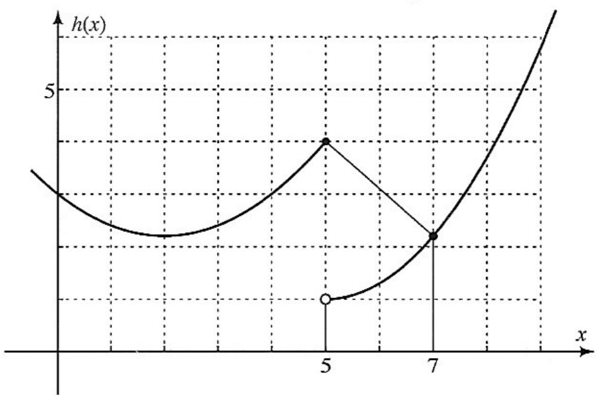
Is h continuous on the closed interval $[5, 7]$? • no

Why is there *no* value $x = c$ in $(5, 7)$ for which $h'(c)$ equals the slope of the secant line?

• The secant line has slope $\dfrac{2.2 - 4}{7 - 2} = -0.9$.

• For x between 5 and 7, the tangent lines have positive slope

• Thus no tangent line has slope -0.9.



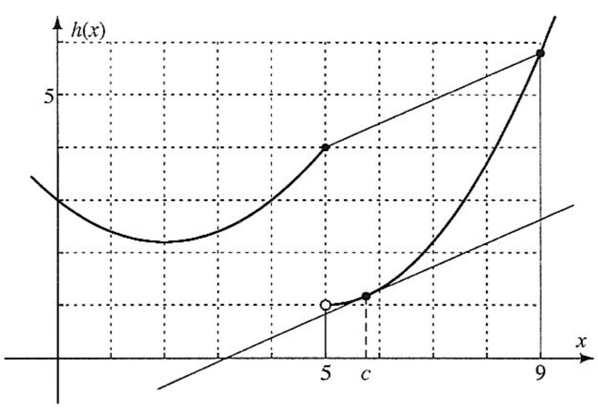
6. The graph below is function h from Problem 5.

Draw a secant line through $(5, h(5))$ and $(9, h(9))$

Is h differentiable on the open interval $(5, 9)$? • yes

Is h continuous on the closed interval $[5, 9]$? • no

There *is* a point $x = c$ in $(5, 9)$ where $h'(c)$ equals the slope of the secant line. Draw the tangent line. Estimate the value of c. • $c \approx 5.75$



7. The graph below is function h from Problem 5

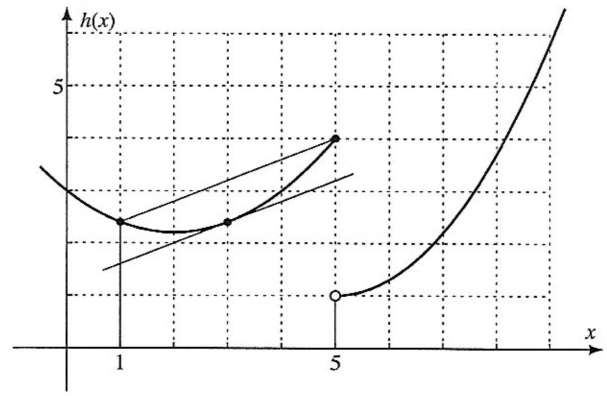
Draw a secant line through $(1, h(1))$ and $(5, h(5))$

Show that there is a point $x = c$ in $(1, 5)$ where $h'(c)$ equals the slope of the secant line. • $c \approx 3$

Is h differentiable on the open interval $(1, 5)$? • yes

Explain why h is continuous on $[1, 5]$, even though there is a step discontinuity at $x = 5$.

• For a closed interval, one-sided continuity at the end points from inside the interval is sufficient for continuity on that interval. And $\lim_{x \to 5^-} h(x) = h(5) = 4$.



8. The mean value theorem states:

If f is differentiable on (a, b) and
f is continuous on $[a, b]$,
then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$ i.e., the secant's slope

For which problem(s) are

the hypotheses and conclusion true? • 1, 2, 4, 7

he hypotheses and conclusion not true? • 3, 5

the conclusion true, but not the hypotheses? • 6

9. The number c is an x-value where the *instantaneous* rate of change equals the *average* ("mean") rate of change. Explain why the hypotheses are **sufficient** conditions for the conclusion, but *not* **necessary** conditions.

• If the hypotheses are true, then the conclusion is true. So the hypotheses "do the job" (sufficient).

• But the conclusion can be true (as in Problem 6) even if the hypotheses are not true. So the hypotheses are not "necessary" to do the job.

10. What did you learn as a result of working this Exploration that you did not know before?

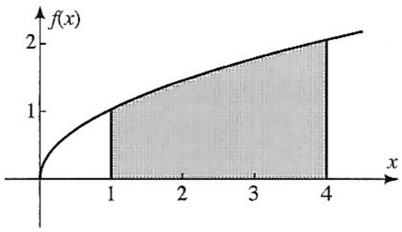
• Answers will vary.

Name ______________________ Group Members: ______________________________

Solutions, Exploration 5-6a: Some Very Special Riemann Sums

Objective: Calculate Riemann sums for given sets of sample points and reach a conclusion about how the sample points were chosen.

The figure shows the graph of $f(x) = x^{1/2}$. In this Exploration you will integrate $f(x)$ from $x = 1$ to 4.



1. Use the fundamental theorem of calculus to find the exact value of the definite integral

$$I = \int_1^4 x^{1/2}\,dx$$

- $= \frac{2}{3}x^{3/2}\Big|_1^4$
- $= \frac{2}{3}4^{3/2} - \frac{2}{3}1^{3/2}$
- $= 4\frac{2}{3}$

2. Find an estimate for the integral I by trapezoidal rule with $n = 3$ subintervals. Write down all the decimal places your calculator will give you. Does this value overestimate or underestimate the actual integral?

- $T_3 = $ 4.64626437...
- Underestimate, by 0.02040229...

3. Find a midpoint Riemann sum for integral I in Problem 1. Use $n = 3$ increments. Does the midpoint sum overestimate or underestimate the actual integral? Which is closer to the actual integral, the trapezoidal sum or the Riemann sum?

- $M_3 = 4.676712395...$
- Overestimate, by 0.0100457...
- The midpoint Riemann sum is closer to the actual integral than the trapezoidal sum.

4. Find a Riemann sum for integral I using the subintervals in Problem 2, but using the following sample points. (k stands for the subinterval number.)

k	$x = c$
1	1.4858425557
2	2.4916102607
3	3.4940272163

How does this sum compare with the exact answer in Problem 1?

- $R_3 = $ 4.666666667 (rounded by calculator)
- Same as the exact answer to 10 significant digits.

5. Find a Riemann sum for I using six subintervals of equal width, and these sample points:

k	$x = c$
1	1.2458051304
2	1.7470136081
3	2.2476804000
4	2.7481034438
5	3.2483958519
6	3.7486100806

How does this sum compare with the exact answer in Problem 1?

- $R_6 = \frac{1}{2}(9.333333333...)$
- $=$ 4.666666667 (rounded by calculator)
- Same as the exact answer to 10 significant digits.

6. Let $g(x) = \frac{2}{3}x^{3/2}$. Find the point in the open interval (1, 1.5) at which the conclusion of the mean value theorem is true for function g. Where have you seen this number in this Exploration?

- $g'(c) = c^{1/2} = \dfrac{g(1.5) - g(1)}{1.5 - 1}$
- $c^{1/2} = 1.116156409...$
- $c = $ 1.24580513..., which is the same as the first sample point in Problem 5.

7. Describe verbally how you can find a Riemann sum for a definite integral that is a *constant*, independent of the number of terms in the sum.

- Choose the sample point in each subinterval a the point where the mean value conclusion is true for the antiderivative function in that interval.

8. What did you learn as a result of doing this Exploration that you did not know before?

- Answers will vary.

Name: ________________ Group Members: ________________________________

Foerster Calculus Solutions, Exploration 5-6b: Proof of the Fundamental Theorem of Calculus

Objective: Prove that a definite integral can be calculated *exactly*, using an indefinite integral.

---

1. Write the definition of $\int_a^b f(x)\,dx$.

   • $\int_a^b f(x)\,dx = \lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n$
   provided both limits exist and are equal.

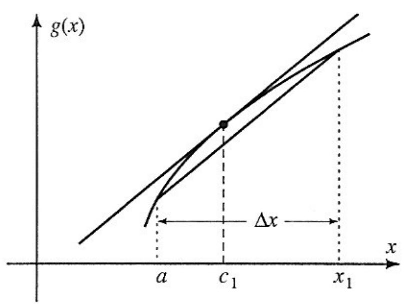
2. Write the definition of $g(x) = \int f(x)\,dx$.

   • $g(x) = \int f(x)\,dx$ if and only if $g'(x) = f(x)$.

3. How can you be sure that the mean value theorem applies to function g?

   • g is differentiable on $[a, b]$, and thus on (a, b), because $g'(x) = f(x)$ for all x in $[a, b]$. g is continuous on $[a, b]$ because differentiability implies continuity.

4. The figure shows function g in Problem 2. Write the conclusion of the mean value theorem as it applies to g on the interval from $x = a$ to $x = x_1$ and illustrate the conclusion on the graph.



   • There is a number c_1 in (a, x_1) such that

   • $g'(c_1) = \dfrac{g(x_1) - g(a)}{\Delta x}$

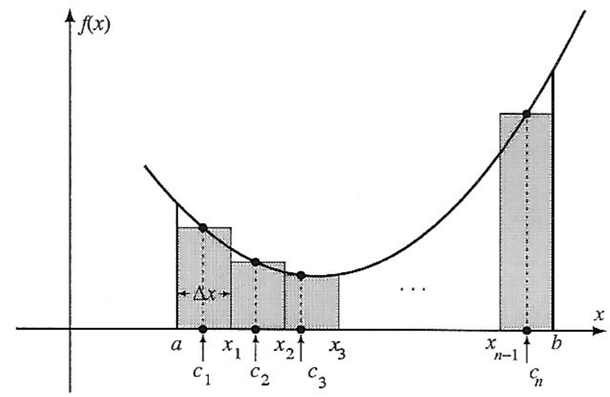
   • See graph. Tangent at c_1 parallels secant line.

5. The figure in the next column shows the graph of $f(x)$ from Problem 2. Let $c_1, c_2, c_3, \ldots, c_n$ be sample points determined by the mean value theorem as in Problem 4. Write a Riemann sum R_n for $\int_a^b f(x)\,dx$ using these sample points and equal Δx values. Show the Riemann sum on the graph.

   • $R_n = f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + \ldots + f(c_n)\Delta x$

   • $R_n = [f(c_1) + f(c_2) + f(c_3) + \ldots + f(c_n)]\Delta x$

   • See graph in next column.



6. By the definition of indefinite integral, $f(c_1) = g'(c_1)$. By the mean value theorem, $g'(c_1) = \dfrac{g(x_1) - g(a)}{\Delta x}$, and so on. By appropriate substitutions, show that R_n from Problem 5 is equal to $g(b) - g(a)$.

   • $R_n = [f(c_1) + f(c_2) + f(c_3) + \ldots + f(c_n)]\Delta x$

   • $R_n = [g'(c_1) + g'(c_2) + g'(c_3) + \ldots + g'(c_n)]\Delta x$

   • $R_n = \left[\dfrac{g(x_1) - g(a)}{\Delta x} + \dfrac{g(x_2) - g(x_1)}{\Delta x} + \dfrac{g(x_3) - g(x_2)}{\Delta x} + \ldots + \dfrac{g(b) - g(x_{n-1})}{\Delta x}\right]\Delta x$

   • $R_n = g(x_1) - g(a) + g(x_2) - g(x_1) + g(x_3) - g(x_2) + \ldots + g(b) - g(x_{n-1})$

   • $R_n = g(b) - g(a)$ (The middle terms "telescope.")

7. R_n from Problem 6 is *independent* of n, the number of increments. Use this fact, and the fact that $L_n \le R_n \le U_n$ to prove that

   $\int_a^b f(x)\,dx = g(b) - g(a)$.

   • Proof: $L_n \le R_n \le U_n$
   • $\lim_{n\to\infty} L_n \le \lim_{n\to\infty} R_n \le \lim_{n\to\infty} U_n$
   • $\lim_{n\to\infty} L_n \le g(b) - g(a) \le \lim_{n\to\infty} U_n$ (lim of const.)
   • $\int_a^b f(x)\,dx \le g(b) - g(a) \le \int_a^b f(x)\,dx$ (def. int.)
   • $\therefore\ \int_a^b f(x)\,dx = g(b) - g(a)$, Q.E.D. (Squeeze thm)

8. The conclusion in Problem 7 is called the **fundamental theorem of calculus**. Show that you understand what it says by using it to find the *exact* value of $\int_1^4 x^{1/2}\,dx$.

   • $\int_1^4 x^{1/2}\,dx = \frac{2}{3}x^{3/2}\Big|_1^4 = \frac{2}{3}4^{3/2} - \frac{2}{3}1^{3/2} = 4\frac{2}{3}$

Name ________________________ Group Members: ____

Solutions, Exploration 5-10c Definite Integral as an Accumulated Rate

Objective: Apply definite integration to a function whose rate of change is known.

The conclusion of the fundamental theorem is

$$\int_a^b f(x)\,dx = g(b) - g(a)$$

Because the function inside the integral sign is the derivative of the function outside, this conclusion can be written just in terms of a function and its derivative. Replacing $g(x)$ with $f(x)$ and $f(x)$ with $f'(x)$ gives this alternate form of the conclusion.

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Solving for $f(b)$ gives

$$f(b) = f(a) + \int_a^b f'(x)\,dx$$

This form of the conclusion says verbally,

> "To find $f(b)$, start at $f(a)$ and add the **accumulated rate of change** of $f(x)$."

1. Demonstrate that you understand this form of the fundamental theorem by finding $f(4)$ if $f(1) = 13$ and $f'(x) = 3x^2$.

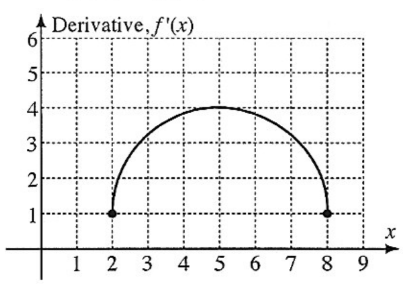
   • $f(4) = 13 + \int_1^4 3x^2\,dx$

   $= 13 + x^3 \big|_1^4$

   $= 13 + (64 - 1)$

   $= \underline{76}$

2. This form of the fundamental theorem is useful even if you cannot find the integral algebraically. The figure shows the semicircular graph of $f'(x)$. Find $f(8)$ if $f(2) = 7$.



   • $f(8) = 7 + \int_2^8 f'(x)\,dx$

   $= 7 + (6 + \frac{1}{2}\pi \cdot 3^2)$

   $= \underline{13 + 4.5\pi}$

3. Find $f(a)$ if $f(b) = 20$ and $\int_a^b f'(x)\,dx = 9$.

   • $20 = f(a) + 9$

   • $\therefore f(a) = 20 - 9 = \underline{11}$

4. **Department Store Problem** – Let $C(x)$ be the number of customers in a particular department store as a function of x hours since opening time. At time $x = 1$ there are 83 customers in the store The table shows $C'(x)$, the rate of change of $C(x)$, (customers/hour in minus customers/hour out) at various times. Estimate the number of customers in the store at time 8 hours. Use an appropriate trapezoidal sum for the integral.

x	Derivative, $C'(x)$
1	200
2	440
5	60
6	–50
8	–160

   • $C(8) = 83 + \int_1^8 C'(x)\,dx$

   $\approx 83 + \frac{1}{2}(1(200 + 440) + 3(440 + 60) + 1(60 - 50) + 2(-50 - 160))$

   $= 83 + 865$

   $= 948$

   • <u>About 948 customers</u> are in the store at time $x = 8$

5. What real-world quanatity does the integral in Problem 4 represent?

   • The integral shows that the number of customers in the store increased by about 865 between times $x = 1$ and $x = 8$ hours. (Mathematically, this is the **accumulated rate of change** of $C(x)$.)

6. What did you learn as a result of doing this Exploration that you did not know before?

   • Answers will vary.

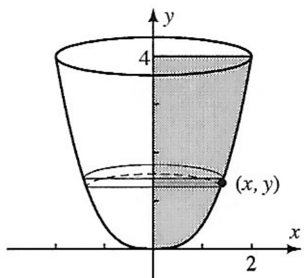
Name ______________________ Group Members: ________________________________

Solutions, Exploration 8-4a: Volumes by Cylindrical Shells

Date: ________

Objective: Find the volume of a solid of revolution by appropriate calculus.

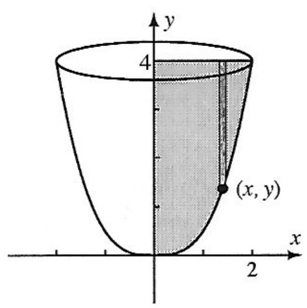
For Problems 1 and 2, the figure shows the region R in Quadrant I bounded by the graph of $y = 0.5x^3$ and the line $y = 4$. The region is rotated about the y-axis to form a solid.

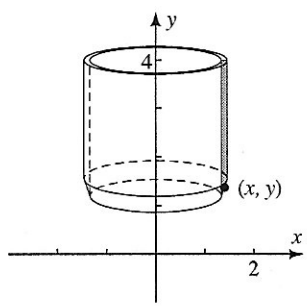


1. The strip shown is perpendicular to the axis of rotation. The sample point (x, y) is on the graph, within the strip. Draw the slice of the figure formed by the strip as it rotates. Then write dV, the volume of the slice, in terms of the sample point.
   - Graph, showing back half of slice.
   - $dV = \pi x^2\, dy$

2. Do the algebra to get dV in terms of one variable. Then find the volume of the entire solid by appropriate integration. You may leave the answer as a multiple of π.
   - $dV = \pi(2y)^{2/3}\, dy$
   - $V = \pi \int_0^4 (2y)^{2/3}\, dy$
   - $V = \underline{9.6\pi} = \underline{30.1592...}$

For Problems 3–4, the figure on the left shows the same solid, but with the strip of the region drawn *parallel* to the axis of rotation. As the region rotates, the strip generates a **cylindrical shell**, as shown on the right.

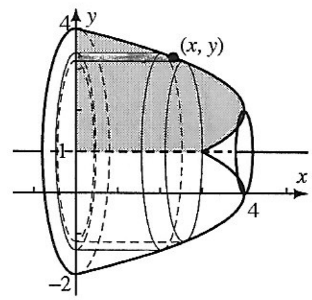




3. Write dV, the volume of the cylindrical shell, in terms of the sample point. Note that the volume equals the circumference of the shell at the sample point times the altitude of the shell at the sample point times the thickness of the shell.
   - $dV = 2\pi x \cdot (4 - y) \cdot dx$

4. Do the algebra to get dV in terms of one variable. Then write an integral equal to the volume of the entire solid. (What are the limits of integration?). Find the volume by doing the integration. If you don't get the same answer as in Problem 2, go back and check your work.
   - $dV = 2\pi x(4 - 0.5x^3)\, dx$
   - $V = 2\pi \int_0^2 x(4 - 0.5x^3)\, dx$
   - Limits of integration are 0 to 2 because the innermost shell is at $x = 0$ and the outermost shell is at $x = 2$.
   - $V = \underline{9.6\pi}$, which agrees with Problem 2

5. The figure shows the solid formed by rotating about the line $y = 1$ the region bounded by the graphs of $x = 4y - y^2$, the y-axis, and the line $y = 1$. Find the volume of the solid by slicing the region parallel to the axis of rotation and using the resulting cylindrical shells.



   - Graph, showing cylindrical shell
   - $dV = 2\pi(y - 1) \cdot x \cdot dy = 2\pi(y - 1)(4y - y^2)\, dy$
   - $V = 2\pi \int_1^4 (y - 1)(4y - y^2)\, dy$
   - $V = \underline{22.5\pi} = \underline{70.6858...}$

6. What did you learn as a result of doing this exploration that you did not know before?
   - Answers will vary.

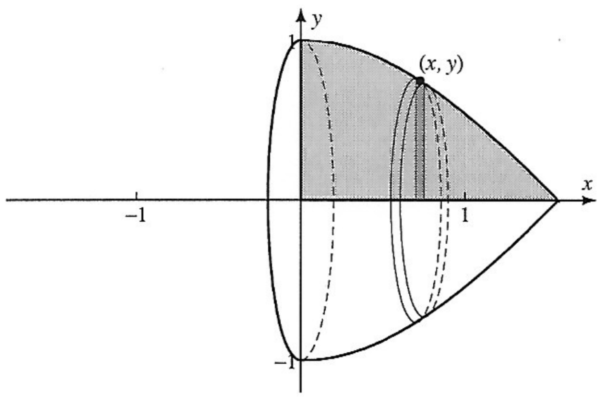
Name ____________________ Group Members: ____________________

Solutions, Exploration 5-9e: Volume Game!

Date:

Objective: Find the volume of various solids by plane slicing.

The figure shows the region under a quarter-cycle of the graph of $y = \cos x$.



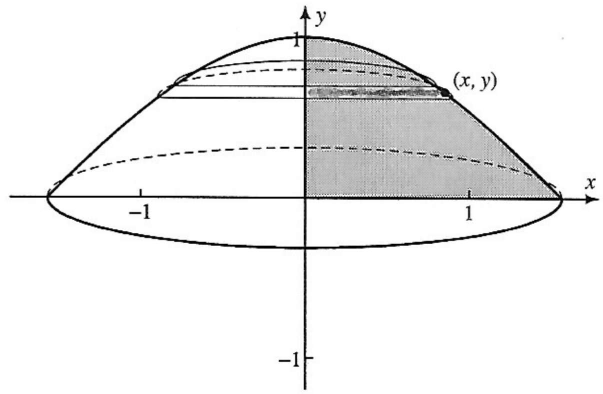
1. Draw a sample point on the graph and a representative slice of the region containing this sample point. Use the results to find the area of the region.

- See figure for strip.
- $dA = \cos x \, dx$
- $A = \int_0^{\pi/2} \cos x \, dx = \underline{1}$ (Surprising!)

2. On the given figure, sketch the solid formed by rotating the region about the x-axis. Show the disk formed as the slice of the region rotates. Find the volume of the solid. Is the answer an "interesting" multiple of π?

- See figure at Problem 1 for disk.
- $dV = \pi y^2 \, dx = \pi \cos^2 x \, dx$
- $V = \pi \int_0^{\pi/2} \cos^2 x \, dx = 0.7853...\pi = \underline{2.4674...}$
- Yes. $V = 0.25\pi^2$

3. On this copy of the figure, sketch the graph of the solid formed by rotating the region about the y-axis. Show a representative slice of the region, the corresponding sample point, and the disk formed as this slice rotates. Find the volume of the solid.

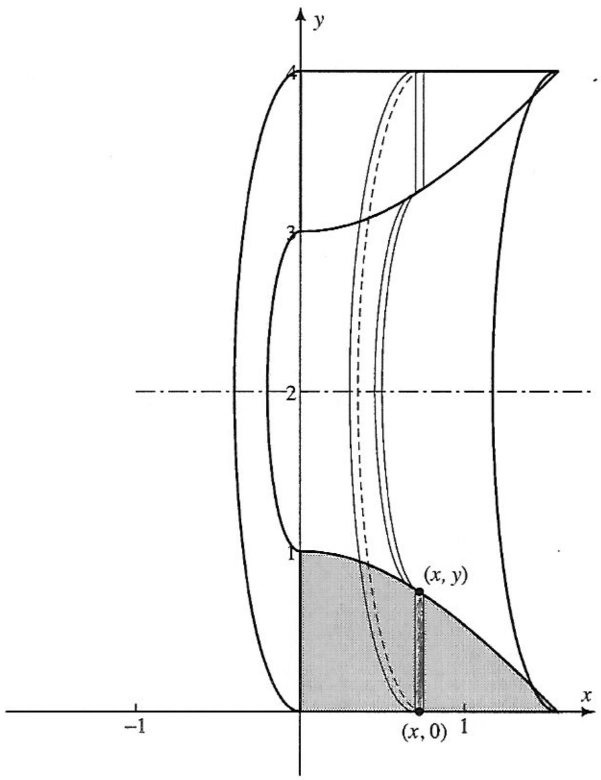


- See figure for solid, strip, and (half)disk.
- $dV = \pi x^2 \, dy = \pi(\cos^{-1} y)^2 \, dy$
- $V = \pi \int_0^1 (\cos^{-1} y)^2 \, dy$

  $= 1.1415...\pi = \underline{3.5864} = \underline{(2 - \pi)\pi}$

(Other side)

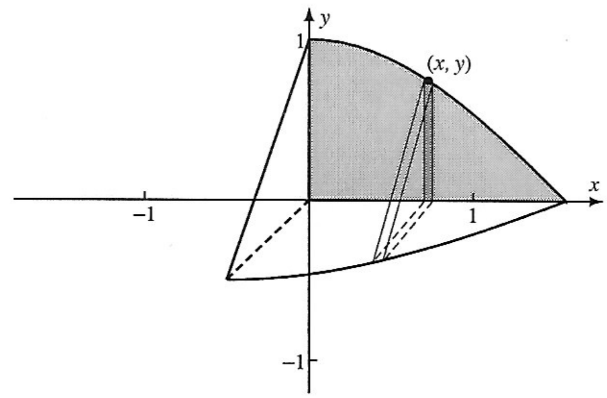
4. On this copy of the figure from the other side, sketch the solid formed by rotating the region about the line $y = 2$. Show a representative slice of the region, the corresponding sample point, and the washer formed as this slice rotates. Find the volume of the solid.



- See figure for half of solid, strip, & half-washer.
- $dV = (\pi(2^2) - \pi(2 - y)^2)\ dx$
- $dV = \pi(4 - (2 - \cos x)^2)\ dx$
- $V = \pi \int_0^{\pi/2} (4 - (2 - \cos x)^2)\ dx$

  $= \underline{3.2146...\pi} = \underline{10.0989...}$

5. The figure shows the region from the other problems in this Exploration. The region forms the base of a solid. Each cross-section perpendicular to the x-axis is an isosceles right triangle with one leg in the base of the solid. Find dV, the volume of the triangular slice of this solid shown in the figure. Then integrate to find the volume of the solid.



- $dV = \frac{1}{2}(y)(y)\ dx$
- $dV = 0.5 \cos^2 x\ dx$
- $V = 0.5 \int_0^{\pi/2} \cos^2 x\ dx = \underline{0.3926...} = \underline{\pi/8}$

6. What did you learn as a result of doing this Exploration that you did not know before?
   - Answers will vary.

Name ______________________ Group Members: ________________________________

Solutions, Exploration 11-1a: Chair Work Problem

Date: ________

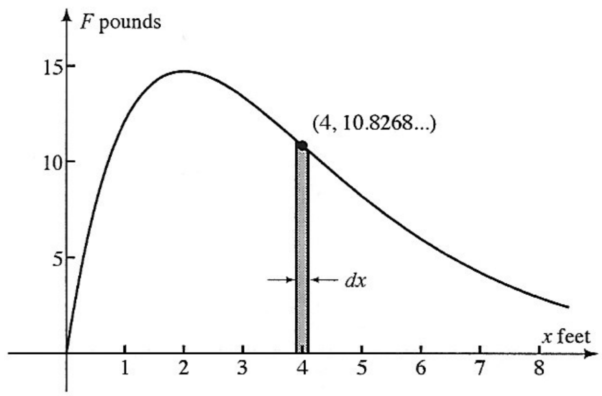
Objective: Find the work done in dragging a chair across the floor.

In physics you learn that work, W, done in moving an object from one place to another is defined to be the force exerted on the object times the distance the object moves. In this Exploration you will calculate the work done by a *variable* force.

Suppose that you push a chair across the floor with a force F in pounds given by the equation

$$F = 20xe^{-0.5x}$$

where x is the distance in feet that the chair has moved from its starting point. The figure shows the graph of F as a function of x.



1. Draw a narrow vertical strip of width dx centered at a sample point on the graph where $x = 4$. Approximately what is the force at any value of x on the graph in this strip? Approximately how much work is done in moving the chair a distance dx feet in this strip?
   - Graph, above.
   - At $x = 4$, $F = 10.8268... \approx 10.8$ pounds
   - $W \approx (10.8268...)(0.2) = 2.1652... \approx 2.2$ ft-lb

2. Write an equation for dW, the work done in moving the chair a distance dx when x is at 4 feet.
   - $dW = F\,dx$
   - $dW = 20xe^{-0.5x}\,dx$

3. Which of the four concepts of calculus would you use to add up all the values of dW as the chair moves from $x = 0$ to $x = 7$, and find the limit of this sum as dx approaches zero and thus the number of strips approaches infinity?
   - Definite integral

4. Find the work done in moving the chair from $x = 0$ to $x = 7$. Why is "foot-pounds" an appropriate name for the units of work?
   - $W = 20\int_0^7 xe^{-0.5x}\,dx$

     $= 20(-2xe^{-0.5x} - 4e^{-0.5x})\Big|_0^7$

     $= 80 - 360e^{-3.5}$

     $= 69.1289...$

     ≈ 69.1 ft-lb

u	dv
x	$e^{-0.5x}$
1	$-2e^{-0.5x}$
0	$4e^{-0.5x}$

   - Force F has the units "pounds" and dx has the units "pounds," so the integrand has the units "(pounds)(feet)," written "foot-pounds.

5. If you continue to push the chair with a force F given by the equation, and the chair continues to move, what limit would the amount of work approach at x approaches infinity? Name the calculus topic you use to answer this question.
   - At an upper limit of integration b, the integral is

     $(-2b - 4)e^{-0.5b}$
   - $\lim_{b\to\infty} (-2b - 4)e^{-0.5b}$

     $= \lim_{b\to\infty} \dfrac{-2b - 4}{e^{0.5b}} \to \dfrac{-\infty}{\infty}$

     $= \lim_{b\to\infty} \dfrac{-2}{0.5e^{0.5b}} \to \dfrac{-2}{\infty}$

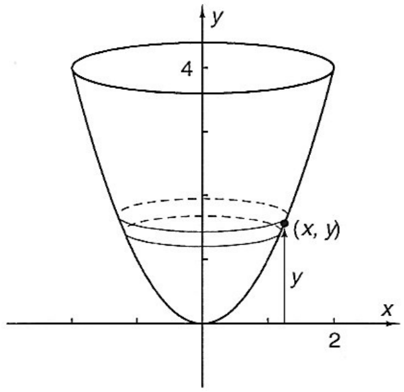
     $= 0$
   - Therefore the work approaches the constant 80 ft-lb

8. What did you learn as a result of doing this exploration that you did not know before?
   - Answers will vary.

Name: ________________ Group Members: ______________________________

Foerster Calculus Solutions, Exploration 11-3: Mass of a Variable-Density Solid Date: ____________

Objective: Find the mass of a solid of revolution if its density varies axially or radially.



1. The figure shows the paraboloid formed by rotating about the y-axis the region above the graph of $y = x^2$, below $y = 4$, and to the right of the y-axis, where x and y are in centimeters. Assume that the density of the solid varies **axially** (in the direction of the axis of the solid), being equal to $3y^{1/2}$ grams per cubic centimeter at a sample point (x, y) in a horizontal disk. Find the mass, dm, of the disk in terms of the sample point. Then use appropriate calculus to find the mass of the entire solid.

   - Slice the region horizontally, so that each point in the slice has approximately the same density as at the sample point (x, y). As the slice rotates, it generates a **disk** with all points having approx. the same density as at the sample point.

   Let dm = mass of a representative slice.

   - Do physics: mass = (density)(volume), so

   $dm = \rho\, dV$

   - Do geometry: Volume of disk = (area)(thickness)

   $dV = (\pi x^2)(dy)$, so

   $dm = 3y^{1/2}(\pi x^2\, dy)$

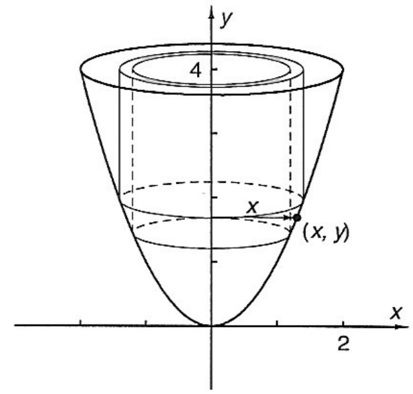
   - Do algebra to get dm in terms of *one* variable. Choose y because the differential is dy.

   $dm = 3\pi y^{3/2}\, dy$

   - Do calculus to add up all the dm s and take the limit. (i.e., integrate):

   $m = 3\pi \int_0^4 y^{3/2}\, dy = \frac{6}{5}y^{5/2}\Big|_0^4$

   $= 38.4\pi = 120.6371... \approx 120.6$ g



2. The figure shows another solid congruent to the solid in Problem 1. The density of this solid varies **radially** (in the direction of the radius), being equal to $x + 5$ grams per cubic centimeter at a sample point (x, y) in a cylindrical shell. Find the mass, dm, of the cylindrical shell in terms of the sample point. Then use appropriate calculus to find the mass of the entire solid.

   - Slice the region vertically, so that each point in the slice has approximately the same density as at the sample point (x, y). As the slice rotates, it generates a **cylindrical shell**, with all points having approximately the same density as at the sample point.

   - Do physics.

   $dm = \rho\, dV$

   - Do geometry: Volume of shell = (area)(thk.), and area = (circumference)(length)

   $dV = (2\pi x)(4 - y)(dx)$, so

   $dm = (x + 5)(2\pi x(4 - y)\, dx)$

   - Do algebra to get dm in terms of one variable.

   $dm = (x + 5)(2\pi x(4 - x^2)dx)$

   $dm = 2\pi(20x + 4x^2 - 5x^3 - x^4)\, dx$

   - Do calculus to add up all the dm s and take the limit. (i.e., integrate):

   $m = 2\pi \int_0^2 (20x + 4x^2 - 5x^3 - x^4)\, dx$

   $m = 48\frac{8}{15}\pi = 152.4719... \approx 152.5$ g

3. What did you learn as a result of doing this Exploration that you did not know before?

   - Answers will vary.